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SCHOOL ALGEBRA
BOOK I

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FROM

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Catharine B. Runkle.
North Hall
Waltham School
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Waltham. Mass.

WENTWORTH-SMITH MATHEMATICAL SERIES

SCHOOL ALGEBRA

BOOK I

BY

GEORGE WENTWORTH

AND

DAVID EUGENE SMITH

GINN AND COMPANY

BOSTON · NEW YORK · CHICAGO · LONDON

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PREFACE

In the preparation of this work the authors have consulted the courses of study in general use in the various states and leading cities of this country, and have considered with great care the syllabi and suggested curricula prepared by the various important associations of teachers of mathematics. They have also studied the papers recently set by the principal examination boards and have taken the judgment of a large number of prominent teachers as to the best selection and arrangement of topics for the first and second years' work in high-school algebra. As a result of this careful investigation, extending over a considerable period of time, they have arranged a work in two books, — the first covering the topics generally agreed upon as suited to the pupil's ninth school year; the second offering a thorough review of the first book and carrying the subject through the Binomial Theorem, Logarithms, and the theory of Complex Numbers.

The plan upon which Book I is arranged is at the same time new and conservative. The first chapter sets before the pupil some of the important uses of algebra, and recognizes the fact that the chief feature that he will meet to-day is the equation applied to the formula. It is the formula that the artisan first meets in his trade journal, that the mechanic needs in reading his manuals, and that the business man will use if he requires algebra at all. It is here that the modern function idea is best brought to the attention of the learner, and here he finds the natural connection between the mensuration that he has studied and the new science that he is beginning. The equation and the formula constitute, therefore, the best introduction to the subject.

Thereafter the work proceeds by important topics, a sufficient number of carefully graded drill problems being given with each to allow the teacher to select and change the material from year to year. The applied problems include as many as possible of the kind met in real life, these necessarily being supplemented by others that have their chief value in the interest they arouse and the drill they give. The topics are those that are found in all standard courses of study extending through Quadratics. The theory and illustrative material of Book I will be found to be better adapted to the powers of the boy and girl of about the age of fourteen than is usually the case. It has been so selected and arranged as to give a good working knowledge of the subject, the more abstract theory and the more difficult problems being postponed to Book II. The chapter on Ratio and Proportion has been planned to permit of its earlier use if desired, and is sufficient for the work on proportion that is required in geometry.

The authors have attempted to set forth, in simple language, the modern idea of function, without carrying the work to an unwarranted extreme. They have presented the graph in the same spirit, introducing it gradually and with the definite purpose of leading the pupil to recognize, through visual aids, the nature of negative numbers and the number and nature of the roots of equations.

Particular attention is called to the Cumulative Review in the Appendix. This furnishes the opportunity for a careful review of all the preceding work at the end of each chapter, and the value of the plan will be apparent to every teacher.

The authors wish to express their thanks to the many teachers who have assisted them either by their valuable suggestions or by their care in the reading of the proof. Any suggestions for further improvement of the work will be gratefully received.

GEORGE WENTWORTH
DAVID EUGENE SMITH

CONTENTS

CHAPTER		PAGE
I.	Introduction. Certain Uses of Algebra	1
II.	Negative Numbers	23
III.	Algebraic Expressions	37
IV.	Addition	41
V.	Subtraction	47
VI.	Multiplication	59
VII.	Division	69
VIII.	Simple Equations with One Unknown Quantity	75
IX.	Special Products and Quotients	89
X.	Factors	97
XI.	Fractions	115
XII.	Simple Fractional Equations	131
XIII.	Simultaneous Simple Equations	145
XIV.	Graphs	169
XV.	Powers and Roots	179
XVI.	Quadratic Equations	211
XVII.	Simultaneous Quadratic Equations	225
XVIII.	Ratio and Proportion	237
APPENDIX		
	Cumulative Review	245
	History of Algebra	292
INDEX		297

TO THE TEACHER

The purpose of the Introduction (Chapter I) to the Wentworth-Smith Algebras, as in the case of most other modern textbooks, is to put the students in the right attitude of mind toward the subject and to lead them to see that algebra has something to do with everyday life. Therefore this Introduction makes clear at the outset many of the uses and applications of algebra. The formula, especially as applied to the equation, is the basis of the work, for it is the formula that the artisan first meets in his trade journal, that the mechanic needs in reading his manual, and that the business man will use if he requires algebra at all.

The exercises in this Introduction need not, however, necessarily be assigned to the class. The teacher may, if desired, utilize them as the basis for talks to the pupils about the new subject upon which they are just entering, and as illustrating the practical value of the equation and the formula in everyday life. An opportunity is thus afforded to show the pupils that algebra is a subject of practical value, thus avoiding the discouragement incident to the older and more formal introduction to this study. In case this plan is followed, the formal assignment of exercises may begin with Chapter II.

The modern demand for making use of the function in algebra is recognized on page 40. This topic is presented in such a simple form that any beginner can master it without the slightest difficulty. Any teacher who wishes to emphasize the subject is thus given an opportunity so to do. On the other hand, since the subsequent work does not, except in two or three examples which may be omitted, depend upon the function notation, the latter need not be given.

It should also be stated, in general, that all good textbooks furnish more material than is needed for the work of almost any class, the purpose being to afford an opportunity for varying the assignments from year to year.

SCHOOL ALGEBRA

BOOK I

CHAPTER I

INTRODUCTION. CERTAIN USES OF ALGEBRA

1. Nature of Arithmetic. In arithmetic we usually represent numbers by figures. If the page of a book is seven inches long and four inches wide, we represent these dimensions by 7 in. and 4 in. respectively. If we wish to find the area, we say that

$$\text{area} = 4 \times 7 \text{ sq. in.} = 28 \text{ sq. in.}$$

2. Nature of Algebra. We might state the process of finding the area of the page as follows:

$$\text{area} = \text{width} \times \text{length},$$

meaning that the number of square inches of area equals the product of the number of inches of width and the number of inches of length. We do this in algebra, often using initial letters, thus:

$$a = w \times l.$$

Usually in algebra, however, we do not indicate multiplication by the sign $\times$. We write wl for $w \times l$, and hence we have

$$a = wl.$$

Arithmetically stated: *The number of units of area of a rectangle equals the product of the number of units of width and the number of units of length.*

Algebraically stated: $a = wl$.

One use of algebra, therefore, is the brief statement of the rules of arithmetic.

3. Arithmetical and Algebraic Statements. A comparison of arithmetical and algebraic statements is seen in the following:

Arithmetic	*Algebra*
If 1 book costs \$2, 3 books will cost $3 \times \$2$.	If 1 book costs d dollars, n books will cost nd dollars.
The cost of any number of books equals the cost of one book multiplied by the given number of books.	If c represents the total cost, then $c = nd$.
If a train travels 35 mi. an hour, in 4 hr. it will travel 4×35 mi., or 140 mi.	If a train travels m miles an hour, in h hours it will travel hm miles.
To find the distance traveled by a train in any number of hours, multiply the number of miles per hour by the number of hours.	If d represents the total distance, then $d = hm$.

4. Formula. A rule stated algebraically, in letters, is called a *formula.*

For example, $c = nd$, and $d = hm$, given above, are formulas.

Algebra enables us to express many of the rules of arithmetic very briefly by formulas.

None of the formulas should be memorized unless it is so stated.

In arithmetic we learn how to find the volume of an ordinary box. We say that the volume is equal to the product of the length, width, and height. In algebra, using initials as before, we represent this by the formula

$$v = lwh.$$

In this case, if $l = 3$, $w = 2$, and $h = 1$, we have

$$v = 3 \times 2 \times 1 = 6.$$

In writing algebraic forms it is not customary to express denominations like feet and inches. In arithmetic we would say that if $l = 3$ in., $w = 2$ in., and $h = 1$ in., v would equal $3 \times 2 \times 1$ cu. in., or 6 cu. in.; but in algebra we omit the inches and cubic inches.

If $l = 4$, $w = 3$, and $h = 2$, we have, as before,

$$v = 4 \times 3 \times 2 = 24.$$

We therefore say that the volume of the box is 24 cu. in.

Exercise 1. Formulas

Examples 1 to 10, oral — Examples 11 to 13, written

1. If a rectangle is 8 in. long and 4 in. wide, how many square inches of area does it contain?

2. If a rectangle has a base of 12 in. and a height of 4 in., what is its area?

3. If a rectangle has a base b inches and a height h inches, what is its area?

4. If a rectangle is l inches long and w inches wide, how many square inches of area does it contain?

5. If 1 yd. of velvet costs \$2, what will 9 yd. cost? If 1 yd. costs d dollars, what will y yards cost?

6. At the rate of 3 mi. an hour, how far will a man walk in 2 hr.? At the rate of m miles an hour, how far will he walk in h hours?

7. Read from the formula $a = mn$ the rule for finding the area of a rectangular field m rods wide and n rods long.

8. Read from the formula $v = pqr$ the rule for finding the volume of a rectangular box p inches long, q inches wide, and r inches deep.

9. Read from the formula $c = nd$ the rule for finding the cost (c) of a number of things when the cost of each (d) is given.

10. Read from the formula $n = c \div d$ the rule for finding the number of articles purchased when the cost of all (c) and the cost of each (d) are given.

11. Write a formula for the average cost (d) of each of n things when they cost c dollars in all.

12. Write a formula for the cost (c) of f feet of iron pipe at n cents a foot. Write the formula so as to express the result in cents; in dollars.

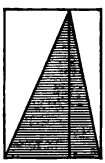
13. If n is any integer, is $2n$ an even number or an odd one? Why? What is the value of $2n$ when $n = 197$? 276? 997?

5. Formula for the Parallelogram. It is usually shown in arithmetic that the area of a parallelogram, like the area of a rectangle, equals the product of its base and height. We may therefore express this in algebraic form thus: $a = bh$.

For the triangle T may be cut off and placed at X, so as to make a rectangle of area bh.

If $b = 5$ and $h = 3$, then $a = bh = 5 \times 3 = 15$. If b and h represent inches, then a represents square inches. When we speak of the product of two lines we mean the product of their numerical values.

6. Formula for the Triangle. It is also shown in arithmetic that the area of a triangle equals half the product of its base and height. This is expressed in algebraic form thus: $a = \frac{1}{2}bh$.



This triangle may be cut as here shown so that it is seen to be half of the rectangle of base b and height h.

If $b = 7$ and $h = 10\frac{1}{2}$, then $a = \frac{1}{2}bh = \frac{1}{2}$ of $7 \times 10\frac{1}{2} = 36\frac{3}{4}$.

Exercise 2. The Parallelogram and Triangle

Examples 1 to 6, oral — Examples 7 to 11, written

1. If $b = 4$ and $h = 3$, what is the value of bh? of $\frac{1}{2}bh$?
2. If $b = 60$ and $h = 5$, what is the value of bh? of $\frac{1}{2}bh$?

Given $a = bh$, find the value of a when:

3. $b = 60$, $h = 11$.
4. $b = 45$, $h = 10$.
5. $b = 20$, $h = 12$.
6. $b = 22$, $h = 10$.

Given $a = \frac{1}{2}bh$, find the value of a when:

7. $b = 48$, $h = 25$.
8. $b = 36$, $h = 19\frac{1}{2}$.
9. $b = 24.8$, $h = 4.75$.
10. $b = 63.2$, $h = 19.65$.

11. A playground is l feet long and w feet wide. Find the area in square feet. What is the number of square feet when $l = 124$ and $w = 62\frac{1}{2}$?

7. Symbols. The following symbols of operation are among the ones that are most commonly used:

	Arithmetic	*Algebra*
Addition	$4 + 3$	$a + b$.
Subtraction	$4 - 3$	$a - b$.
Multiplication	4×3	$a \times b$, $a \cdot b$, or ab.
Division	$4 \div 3$	$a \div b$, $a : b$, or $\frac{a}{b}$.

In algebra the fraction form is the most common one for expressing the division of one quantity by another.

Second power (square) 5^2 means 5×5; a^2 means aa.

The second power, or square, of a number means the product arising from taking the number twice as a factor.

Third power (cube) 5^3 means $5 \times 5 \times 5$; a^3 means aaa.

The third power (cube) of a number means the product arising from taking the number three times as a factor, and so for other powers.

Square root	$\sqrt{4}$ means the square root of 4, or one of the two equal factors of 4.	$\sqrt{a}$ means the square root of a, or one of the two equal factors of a.

If $a = 3$, then $\sqrt{a}$ can be expressed only approximately as a decimal fraction, and similarly for $a = 5$, $a = 7$, and so on.

Cube root	$\sqrt[3]{27}$ means the cube root of 27, or one of the three equal factors of 27.	$\sqrt[3]{b}$ means the cube root of b, or one of the three equal factors of b.

It is customary to use the symbol $\therefore$ for "therefore."

8. Monomial. An algebraic expression in which the parts are not separated by the signs + or − is called a *monomial.*

Thus a, $3ab$, a^2, and $\sqrt{a}$ are monomials. By an algebraic expression we mean any expression in which letters are used to represent some or all of the numbers, like $a + b$, $3a$, $2n + 1$.

9. Polynomial. An algebraic expression consisting of two or more monomials is called a *polynomial.*

Thus $a + b$, $3a - 4b$, $2n + 1$, and $a - b + c$ are polynomials

10. Terms of a Polynomial. The monomials that make up a polynomial are called the *terms* of the polynomial.

A polynomial is called a *binomial* if it has only two terms, and a *trinomial* if it has only three terms.

The terms of the binomial $a + 3b$ are a and $3b$.
The terms of the trinomial $a + 3b + c$ are a, $3b$, and c.

11. Symbols of Aggregation. Symbols that indicate that certain terms are to be treated as one number or one quantity are called *symbols of aggregation.*

The most common of these are the parentheses, brackets, and bar. Others will be given when needed. Thus $2 \times (5 + 7)$, $2 \times [5 + 7]$, or $2 \times \overline{5 + 7}$ means that 5 and 7 are to be added before we multiply by 2; each is read "twice the sum of 5 and 7," and equals 24. The expression $\sqrt{3 + 6}$, having the bar extending from the root sign over $3 + 6$, means that 3 and 6 are to be added before we extract the square root.

12. Order of Operations. Unless some symbols of aggregation direct otherwise, in an algebraic expression the operations are performed in the following order:

1. *Powers and roots.*

2. *Multiplications and divisions, and these are taken in the order in which they occur.*

3. *Finally, additions and subtractions, and these may be taken in the order in which they occur or in any other order.*

Thus $\sqrt{4^3}$ means that we first cube 4, the result being 64; then take the square root of 64. It will be seen that we get the same answer if we first extract the square root of 4 and then cube the result.

In the case of $a^2b + c \div d$ we first square a; we then take the product of a^2 and b; we then divide c by d; and finally we add the results. Thus if $a = 2$, $b = 3$, $c = 10$, and $d = 5$, we have

$$2^2 \times 3 + 10 \div 5 = 4 \times 3 + 10 \div 5 = 12 + 2 = 14.$$

Similarly, we have the following:

$2\sqrt{4} + 6 \div 2 \times 3 = 2 \times 2 + 6 \div 2 \times 3 = 4 + 3 \times 3 = 4 + 9 = 13$;

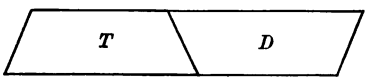
$5 \times 5 - 2^2 \times 3 + 12 \div 2 + 2 = 25 - 4 \times 3 + 6 + 2 = 21$;

$\sqrt{3 + 6} + 15 \div 5 - 2 + 7 \times 6 = 3 + 3 - 2 + 42 = 46$;

$24 \div 12 \times 2^2 - 5 + 2 = 24 \div 12 \times 4 - 5 + 2 = 2 \times 4 - 5 + 2 = 5.$

13. Formula for the Trapezoid. It is usually shown in arithmetic that the area of a trapezoid equals half the product of the sum of the two parallel sides multiplied by the height.

For a trapezoid D, equal to the given trapezoid T, may be turned over and put down by the side of T, as here shown. The whole figure, or twice the trapezoid, then equals a parallelogram whose base is the sum of the parallel sides of the trapezoid. The trapezoid is therefore half this parallelogram.



We indicate this in algebraic form as follows:

$$a = \tfrac{1}{2}(b + b')\,h,$$

where a is the area, b and b' ("b prime") are the two parallel sides, usually called the *bases*, and h is the height.

The parentheses show that b and b' are to be added before being multiplied by h.

Thus if $b = 6$, $b' = 5$, and $h = 4$, we have $a = \tfrac{1}{2}(b + b')\,h = \tfrac{1}{2}(6 + 5) \times 4 = \tfrac{1}{2}$ of $11 \times 4 = 22$.

Exercise 3. The Trapezoid

Examples 1 to 7, oral — Examples 8 to 13, written

1. Find the value of $b + b'$, when $b = 7$, $b' = 9$; when $b = 7$, $b' = 9\tfrac{1}{2}$; when $b = 7\tfrac{3}{4}$, $b' = 9\tfrac{1}{2}$.

2. Find the value of $(b + b')\,h$, when $b = 4$, $b' = 5$, $h = 3$.

Given $a = \tfrac{1}{2}(b + b')\,h$, find the value of a when:

3. $b = 6$, $b' = 4$, $h = 3$.
4. $b = 9$, $b' = 5$, $h = 7$.
5. $b = 10$, $b' = 8$, $h = 6$.
6. $b = 13$, $b' = 7$, $h = 7$.
7. $b = 19$, $b' = 5$, $h = 4$.
8. $b = 24$, $b' = 9.5$, $h = 7.2$.
9. $b = 34$, $b' = 16$, $h = 19$.
10. $b = 38$, $b' = 9.8$, $h = 8.6$.
11. $b = 5.9$, $b' = 3.4$, $h = 1\tfrac{1}{2}$.
12. $b = 4\tfrac{3}{4}$, $b' = 3\tfrac{1}{4}$, $h = 2\tfrac{1}{2}$.

13. A playground is in the form of a trapezoid, with bases x and y, and with height z. What is the area? How many square rods are there in the playground if $x = 30$ rd., $y = 26$ rd., and $z = 24$ rd.? if $x = 34$ rd., $y = 28$ rd., and $z = 30$ rd.?

14. Evaluation of an Expression. The substitution of numerical values for the letters in an algebraic expression, and the reduction of the result to simplest form, is called the *evaluation* of the expression.

Thus to evaluate $2n$ for $n = 5$, substitute 5 for n and we have $2 \times 5 = 10$.

Exercise 4. Evaluation of Expressions

Examples 1 to 7, oral — Examples 8 to 20, written

1. What are the terms of the binomial $a + b$? Evaluate it for $a = 1$ and $b = 2$.

2. What are the terms of the trinomial $x + y + z^2$? Evaluate it for $x = 1$, $y = 1$, $z = 1$.

3. In the expression $4 \times (5 + 2)$, which operation is performed first? Which next? What is the result? What is the value of $4 \times 5 + 2$? What is the value of $4 \times 5 + 2 \times 3 \div 3$?

Evaluate the following expressions:

4. $a = b^2$; $b = 3$.
5. $a = bh$; $b = 12$, $h = 4$.
6. $a = bh$; $b = 22$, $h = 6$.
7. $a = \frac{1}{2}bh$; $b = 22$, $h = 10$.
8. $2x^2 + 3x + 1$; $x = 10$.
9. $3x^3 + 4x^2 + 2x + 3$; $x = 10$.
10. $a + 6b$; $a = 7$, $b = 1$.
11. $(a + b)^2$; $a = 1$, $b = 2$.
12. $(a - b)^2$; $a = 7$, $b = 2$; also $a = 10$, $b = 5$.
13. $a^2 - 2ab + b^2$; $a = 7$, $b = 2$; also $a = 10$, $b = 5$.
14. $a^2 - b^2$; $a = 5$, $b = 3$; also $a = 2$, $b = 1$.
15. $(a + b)(a - b)$; $a = 5$, $b = 3$; also $a = 2$, $b = 1$.
16. $(a + b)^3$; $a = 4$, $b = 3$; also $a = 2$, $b = 10$.
17. $a^3 + 3a^2b + 3ab^2 + b^3$; $a = 4$, $b = 3$; also $a = 3$, $b = 4$.
18. $(a - b)^3$; $a = 23$, $b = 4$; also $a = 49$, $b = 43$.
19. $\sqrt{a^2 + 2ab + b^2}$; $a = 4$, $b = 2$; also $a = 5$, $b = 3$; $a = 7$, $b = 1$.
20. $\sqrt{a^2 - 2ab + b^2}$; $a = 5$, $b = 1$; also $a = 5$, $b = 3$; $a = 6$, $b = 5$.

Exercise 5. Writing Algebraic Expressions

Examples 1 to 14, oral — Examples 15 to 22, written

1. Read $4 + 7$; $a + b$; $3a + b$; $3a + 5b$.

2. Read $9 - 5$; $a - b$; $7a - b$; $7a - 4b$.

3. Read 2×7; $a \times b$; ab; $3ab$; $a(b + c)$; $x(a + b + c)$.

4. Read $9 \div 3$; $a \div 3$; $a \div b$; $3a \div b$; $9a \div b$; $9a \div 7$.

5. Read $2n$; $2n + 1$; $2n - 1$. What is the value of each when $n = 3$? when $n = 5$? when $n = 10$?

6. Read $\frac{1}{2}n$; $\frac{1}{3}n$; $\frac{2}{3}n$; $\frac{3}{4}n$. What is the value of each when $n = 12$? when $n = 24$? when $n = 10$?

7. How many feet in a yard? in 5 yd.? in n yards?

8. How many inches in a foot? in 2 ft.? in n feet?

9. How many ounces in a pound? in 10 lb.? in n pounds?

10. If 1 yd. of cloth costs $20¢$, what will 3 yd. cost? n yards?

11. If 1 lb. of tea costs $\$\frac{1}{2}$, what will 4 lb. cost? n pounds?

12. Read 2 ft.; 2 yd.; $2n$; $2x$; 2 mi.; $2m$; 2 A.; $2a$.

13. Read 3 ft. 2 in.; $3x + 2y$; $3a + 2b$; $\$3 + 2¢$.

14. Read 7 ft. $-$ 2 in.; 7 yd. $-$ 2 ft.; $\$7 - 2¢$; $7x - 2y$.

15. Write the sum of 2 and y; of x and y; of $2x$ and y.

16. Indicate the subtraction of y from 7; of y from x.

17. Write the products: 2 times x; 2 times y; x times y.

18. Write the quotient of x divided by 2, both as a fraction and with some other sign of division.

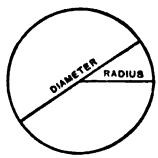
19. In the same two ways write the quotient of $2x$ divided by 3; of $2x$ divided by y; of $2x$ divided by $3y$.

20. Write the sum of $5x^2$, $7x$, and 2. What is its value when $x = 10$? What is its value when $x = 20$?

21. Write the sum of $7x^3$, $2x^2$, $3x$, and 6. What is its value when $x = 10$? What is its value when $x = 20$?

22. Write the sum of $2x^4$, $3x^3$, $3x^2$, $7x$, and 5. What is its value when $x = 10$? What is its value when $x = 1$?

15. Formula for the Circumference. It is usually shown in arithmetic that the circumference of a circle equals nearly 3.1416 × the diameter. In mathematics the number 3.1416 −, or 3.14159 +, which is nearly $3\frac{1}{7}$, is represented by the Greek letter π (pronounced *pī*). We may therefore express this law as follows:



$$c = \pi d,$$

where c stands for circumference, d for diameter, and π for nearly $3\frac{1}{7}$, or nearly 3.1416.

Since the diameter equals twice the radius, we may write $2r$ for d, and have $c = \pi \times 2r$, or $c = 2\pi r$.

Thus if $d = 7$, and we take $3\frac{1}{7}$, or $\frac{22}{7}$, as the value of π, we have

$$c = \pi d = 3\tfrac{1}{7} \times 7 = 22.$$

If $r = 5$, we have $c = 2\pi r = 2 \times 3\frac{1}{7} \times 5 = 31\frac{3}{7}$.

Exercise 6. The Circumference

Examples 1 to 4, oral — Examples 5 to 21, written

1. If $\pi = 3\frac{1}{7}$ and $d = 7$, find the value of πd.
2. If $\pi = 3\frac{1}{7}$ and $r = 3\frac{1}{2}$, find the value of $2r$; of $2\pi r$.

Given $c = \pi d = 2\pi r$, and taking $\pi = 3\frac{1}{7}$, find c when:

3. $d = 14$.
4. $d = 21$.
5. $d = 3.5$.
6. $d = 4.9$.
7. $r = 77$.
8. $r = 3.5$.
9. $r = 6.3$.
10. $r = 9.1$.

Taking $\pi = 3.1416$, find c when:

11. $d = 10$.
12. $d = 20$.
13. $d = 50$.
14. $d = 25$.
15. $r = 30$.
16. $r = 40$.
17. $r = 2\frac{1}{2}$
18. $r = 7.5$.

19. A boy measures the diameter of his bicycle wheel and finds it to be 28 in. What is the circumference?

20. A workman measures the diameter of a steel shaft and finds it to be $3\frac{7}{16}$ in. What is the circumference?

21. If you describe a circle with a radius of 12 in., how many inches will there be in the circumference?

16. Formula for the Area of a Circle. It is usually shown in arithmetic that

1. The area of a circle equals half the product of the circumference and radius; that is, $a = \frac{1}{2}cr$.

2. The area of a circle equals π times the square of the radius; that is, $a = \pi r^2$.

Thus if $r = 10$, we have $a = \pi r^2 = 3.1416 \times 10^2 = 314.16$, taking 3.1416 for π. If r represents the number of feet, the area is in square feet; if r represents the number of inches, the area is in square inches.

Exercise 7. Area of a Circle

Examples 1 to 4, oral — Examples 5 to 23, written

1. If $r = 1$ and $c = 6\frac{2}{7}$, what is the value of cr? of $\frac{1}{2}cr$?
2. If $r = 10$ and $c = 63$, what is the value of cr? of $\frac{1}{2}cr$?
3. If $r = 1$ and $\pi = 3\frac{1}{7}$, what is the value of r^2? of πr^2?
4. If $r = 2$ and $\pi = 3\frac{1}{7}$, what is the value of r^2? of πr^2?

Given $a = \frac{1}{2}cr$, find a when:

5. $r = 5$, $c = 31.416$.
6. $r = 10$, $c = 62.832$.
7. $r = 20$, $c = 125.664$.
8. $r = 2\frac{1}{2}$, $c = 15.708$.
9. $r = 25$, $c = 157.08$.
10. $r = 50$, $c = 314.16$.

Given $a = \pi r^2$, and taking $\pi = 3\frac{1}{7}$, find a when:

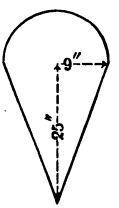
11. $r = 7$.
12. $r = 3\frac{1}{2}$.
13. $r = 14$.
14. $r = 2.1$.
15. $r = 2.8$.
16. $r = 35$.
17. $r = 4.9$.
18. $r = 7.7$.

Given $a = \pi r^2$, and taking $\pi = 3.1416$, find a when:

19. $r = 5$.
20. $r = 10$.
21. $r = 20$.

22. How many square feet in the area of a circle whose radius is 2 ft.? (Take $\pi = 3.1416$.)

23. How many square inches are there in the area of this kite? (Take $\pi = 3\frac{1}{7}$. Also find the result when 3.1416 is taken for π.)



Exercise 8. Formulas of Measurement

Examples 1 to 7, oral — Examples 8 to 12, written

1. It is usually shown in arithmetic that the volume (v) of a cylinder equals the product of the base (b) and height (h). Express this in a formula.

2. It is also shown that the volume equals the product of the height and π times the square of the radius of the base. Express this in a formula.

Express the following statements in formulas:

3. The volume (V) of a cube equals the third power of an edge (e).

4. The entire surface (S) of a cube equals six times the second power of an edge (e).

5. The lateral (or side) area of a cylinder equals the product of the circumference and height. (Use L for lateral area, c for circumference, and h for height.)

6. The lateral area of a cylinder equals 3.1416 times the diameter multiplied by the height.

7. The lateral area of a cylinder equals 3.1416 times the diameter multiplied by the height. (Use d for diameter, and in general use the initial letter for a word in a formula.)

8. The volume of a prism equals the product of the base and height. Evaluate for $b = 17.5$, $h = 8.5$.

9. The volume of a cylinder equals the product of the base and height. Evaluate for $b = 23.2$, $h = 14.3$.

10. The volume of a cylinder divided by the base equals the height. Evaluate for $V = 28.8$, $b = 2.4$.

11. The volume of a pyramid equals one third the product of the base and height. Evaluate for $b = 16.4$, $h = 8.4$.

12. The volume of a pyramid divided by one third the height equals the base. Evaluate for $V = 7.5$, $h = 4.5$.

Exercise 9. Commercial Formulas

Examples 1 to 4, oral — Examples 5 to 14, written

1. At 6% a year, what is the interest on \$200 for 1 yr.?

2. If r is the rate for 1 yr., what is the interest on p dollars for 1 yr.? for 2 yr.? for $2\frac{1}{2}$ yr.?

3. At 6% a year, what is the interest on \$200 for $2\frac{1}{2}$ yr.?

4. If r is the rate for 1 yr., what is the interest on p dollars for t years?

5. If the list price of some goods is \$575 and a discount of 10% is allowed, what is the discount? What is the net price?

6. If the list price of some goods is l dollars, and the rate of discount is r, what is the discount? Write a formula for n, the net price.

7. If some stock is selling 7% below par, what will be the cost of \$1500 worth, par value, brokerage not considered?

8. If some stock is selling at $r\%$ below par, what will be the cost of p dollars' worth, par value, brokerage not considered?

9. What is the amount of principal and interest of \$1700 for 2 yr. at 5%?

10. What is the amount of principal and interest of p dollars for t years, the rate of interest being r?

11. A trade price list gives the cost in dollars per foot of sewer pipe of diameter d inches, as follows: $c = 0.004\,d^2 + 0.14$. Find the cost of 300 ft. of 16-inch pipe.

12. From the formula of Ex. 11, find the cost of half a mile of 18-inch pipe.

13. From the formula of Ex. 11, find the cost of 30 rd. of $8\frac{1}{2}$-inch pipe.

14. If the amount of d dollars on interest for n years at $r\%$ is $d(1 + nr\%)$, what is the amount of \$750 for $3\frac{1}{2}$ yr. at 5%?

Exercise 10. Shop Formulas

Examples 1 to 4, oral — Examples 5 to 16, written

1. Given $b = 7$, $h = 9$, what is the value of bh?

2. Given $b = 12$, $h = 20$, what is the value of $\frac{1}{2}bh$?

3. Given $r = 10$, what is the value of r^2? of r^3?

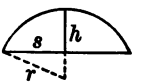
4. Given $r = 1$, $\pi = 3\frac{1}{7}$, what is the value of πr^2?

5. The diameter of a steel shaft of circumference c is $d = \frac{c}{\pi}$. Evaluate to two decimal places for $c = 10$, $\pi = 3\frac{1}{7}$.

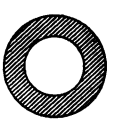
6. A bar of metal has for cross section an equilateral triangle each side of which is s. The area of the cross section is given by the formula $a = \frac{1}{4}s^2\sqrt{3}$. Find to two decimal places the area of the cross section when $s = 7$.

7. The foreman of a shop reads in his book of instruction that the safe load (l) in pounds that can be hoisted by a rope c inches in circumference is given by the formula $l = 100\,c^2$. How many pounds can he safely allow for a rope that is $2\frac{1}{4}$ in. in circumference?

8. A carpenter wishes to put up a circular arch of height h and span $2s$. It is necessary to find the radius of the circle so that he may make his pattern. He knows that $r = \frac{s^2 + h^2}{2h}$. Find the radius, given $h = 2$, $s = 4$.



9. The area of a circle with radius a being πa^2, and of one with radius b being πb^2, the area of the ring between the two circumferences is $\pi a^2 - \pi b^2$. What is the area of the ring if $a = 7$, $b = 4$, and $\pi = 3\frac{1}{7}$?



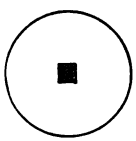
10. A workman needs to find the area of the metal in a cross section of iron pipe of which the exterior diameter is 10 in. and the interior diameter 9 in. Find this from the formula $\pi a^2 - \pi b^2$, where a and b are radii (semidiameters), and $\pi = 3\frac{1}{7}$.

11. In the shop a workman has to cut a circular hole in a circular iron plate. The diameter of the plate is 12 in. and the diameter of the hole is 7 in. The formula for the area of a circle in terms of the diameter is $\frac{1}{4}\pi d^2$. Required the area of the circular plate left after cutting the hole. Take $\pi = 3\frac{1}{7}$.

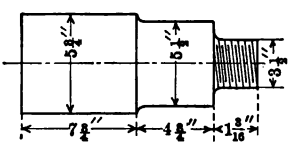
12. On the roof of a shop is a water tank in which the surface of the water is 40 ft. above the ground, and from the tank a pipe leads down through the shop. The owner wishes to find how strong this pipe must be so that it will not burst. He reads in a book on water pipes that the formula for the pressure of water in pounds on every square inch, the surface of the water being h feet above the ground, is denoted by the formula $p = 62\frac{1}{2}\,h + 144$. What is the pressure on the pipe at the level of the ground?

13. Given $v = \pi r^2 h$ as the formula for the volume of a cylinder, find the weight of a steel shaft 6 ft. long and 2 in. in diameter, a cubic inch of steel weighing 0.283 lb.

14. This circular plate has a radius of r inches. A square hole s inches on a side is cut in the plate. What is the area of the remaining part?



15. A machinist is making a "crank pin" (a kind of bolt) for an engine, according to this drawing. He considers it as weighing the same as three steel cylinders having the diameters and lengths in inches as here shown, where $7\frac{3}{4}''$ means $7\frac{3}{4}$ in. He has this formula for the weight (w) in pounds of a steel cylinder where d is the diameter and l is the length in inches: $w = 0.07\,\pi d^2 l$. Taking $\pi = 3\frac{1}{7}$, find the weight of the pin.



16. The volume of a cylinder being represented by $v = \pi r^2 h$, what will the water in a cylindrical water tank weigh when $r = 9$ ft., $h = 10$ ft., $\pi = 3\frac{1}{7}$, and the weight of 1 cu. ft. of water is $62\frac{1}{2}$ lb.?

17. Equation. An expression of equality between two numbers or quantities is called an *equation.*

For example, $x + 5 = 7$. This means that some number increased by 5 equals 7. Evidently this number is 2.

In this equation x is usually called the *unknown quantity.* It is really a number to be determined. Unknown quantities are usually represented by x, y, or z, or else by initial letters, such as p for pounds or d for dollars.

In this equation $x + 5$ is called the *first member* and 7 is called the *second member.*

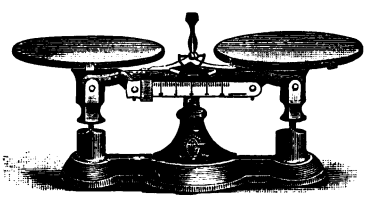
Exercise 11. Using Subtraction

Examples 1 to 10, oral — Examples 11 to 18, written

1. What number increased by 4 equals 7? In the equation $n + 4 = 7$, what is the number n?

2. What weight increased by 5 lb. equals 11 lb.? In the equation $p + 5 = 11$, what is the value of p?

3. What number increased by 7 equals 15? What is the value of x in the equation $x + 7 = 15$?



4. If these scales balance when we place $x + 6$ oz. on one side and 15 oz. on the other side, how much will be left on each of the sides if we take 6 oz. from each? What is the value of x?

5. If we take 10 oz. from one side, how much must be taken from the other side to keep the scales balanced?

6. If equals are subtracted from equals, what can be said of the remainders?

Find the value of x in the following:

7. $x + 9 = 21$.	**11.** $x + 1.7 = 9.1$.	**15.** $x + 9\frac{1}{2} = 10\frac{1}{8}$.
8. $x + 19 = 21$.	**12.** $x + 5.7 = 9.2$.	**16.** $x + 3\frac{1}{4} = 4\frac{1}{2}$.
9. $x + 19 = 49$.	**13.** $x + 0.76 = 0.93$.	**17.** $x + 5\frac{2}{3} = 7\frac{3}{4}$.
10. $x + 29 = 79$.	**14.** $x + 0.77 = 0.95$.	**18.** $x + 9\frac{7}{8} = 10\frac{3}{4}$.

Exercise 12. Using Division

Examples 1 to 19, oral — Examples 20 to 29, written

1. If twice a certain number equals 16, what is the number? If $2n = 16$, what is the value of n?

2. If 5 times a certain number equals 35, what is the number? If $5n = 35$, what is the value of n?

3. At 6¢ each, how many oranges can I buy for 48¢? If $6n = 48$, what is the value of n?

4. At 12¢ a yard, how many yards of ribbon can I buy for 48¢? If $12r = 48$, what is the value of r?

5. At 40¢ a dozen, how many dozen eggs can I buy for \$1.60? If $40e = 160$, what is the value of e?

6. If I put equal weights on the two sides of some scales like those shown on page 16, then $\frac{1}{3}$ of the weight on one side will just balance what part of the weight on the other side?

7. If equals are divided by equals, what can be said of the quotients?

Find the value of x in the following:

8. $2x = 6$.	14. $7x = 21$.	20. $13x = 117$.
9. $3x = 9$.	15. $9x = 63$.	21. $17x = 119$.
10. $3x = 6$.	16. $7x = 91$.	22. $16x = 128$.
11. $4x = 8$.	17. $4x = 72$.	23. $1.9x = 11.4$.
12. $5x = 5$.	18. $5x = 75$.	24. $2.3x = 25.3$.
13. $6x = 6$.	19. $6x = 72$.	25. $0.27x = 0.324$.

26. What number multiplied by 15 equals 240?

27. What number multiplied by 17 equals 221?

28. A man paid \$780 for a certain number of cattle at \$65 a head. How many did he buy?

29. A dealer paid \$10,325 for some automobiles at an average price of \$1475 each. How many did he buy?

Exercise 13. Using Addition

Examples 1 to 13, oral — Examples 14 to 30, written

1. What must be added to 3 to make 8? to $8 - 5$ to make 8? to $x - 5$ to make x? to $n - 5$ to make n?

2. If $x - 5 = 20$, what must be added to these equals to make the first member x? What does the second member then become? What is the value of x?

3. If $n - 7 = 31$, what must be added to these equals to make the first member n? What is the value of n?

4. If $n - 9 = 35$, how will you proceed to find the value of n? What is the value of n?

5. If 6 is subtracted from a certain number, the result is 12. What is the number?

6. If equals are added to equals, what can be said of the results?

Find the value of x in the following:

7. $x - 1 = 7$.	**14.** $x - 27 = 72$.	**21.** $x - 2\frac{1}{2} = 3\frac{1}{2}$.
8. $x - 2 = 7$.	**15.** $x - 34 = 62$.	**22.** $x - 2\frac{1}{4} = 5\frac{3}{4}$.
9. $x - 4 = 6$.	**16.** $x - 71 = 89$.	**23.** $x - 6\frac{2}{3} = 7\frac{2}{3}$.
10. $x - 5 = 8$.	**17.** $x - 8.7 = 8.7$.	**24.** $x - 4\frac{7}{8} = 5\frac{3}{4}$.
11. $x - 6 = 9$.	**18.** $x - 4.9 = 5.7$.	**25.** $x - 3\frac{4}{5} = 8\frac{1}{2}$.
12. $x - 7 = 3$.	**19.** $x - 0.78 = 0.35$.	**26.** $x - 6\frac{2}{3} = 4\frac{1}{4}$.
13. $x - 8 = 5$.	**20.** $x - 0.87 = 0.63$.	**27.** $x - 9\frac{3}{4} = 8\frac{3}{8}$.

28. If 59 is subtracted from a certain number, the result is 59. What is the number? Prove it.

29. If 279 is subtracted from a certain number, the result is 643. What is the number? Prove it.

30. A book has a certain number of pages. After we have read 169 pages there are 138 more to be read. How many pages are there in the book?

Exercise 14. Using Multiplication

Examples 1 to 12, oral — Examples 13 to 28, written

1. By what must $\frac{1}{2}$ in. be multiplied to make 1 in.?

2. By what must $\frac{1}{2}x$ be multiplied to make x?

3. By what must $\frac{3}{4}$ be multiplied to make 3?

4. By what must $\frac{3}{4}x$ be multiplied to make x?

5. By what must both members of the equation $\frac{x}{4} = 7$ be multiplied to give the value of x? What is the value of x?

6. If $\frac{x}{3} = 9$, what is the value of x? Prove it.

7. If equals are multiplied by equals, what can be said of the results?

Find the value of x in the following:

8. $\frac{x}{2} = 9.$

9. $\frac{x}{3} = 7.$

10. $\frac{x}{4} = 8.$

11. $\frac{x}{5} = 5.$

12. $\frac{x}{6} = 4.$

13. $\frac{x}{11} = 10.$

14. $\frac{x}{17} = 16.$

15. $\frac{x}{13} = 14.$

16. $\frac{x}{15} = 12\frac{1}{3}.$

17. $\frac{x}{16} = 14\frac{3}{4}.$

18. $\frac{x}{2.5} = 26.$

19. $\frac{x}{2.7} = 32.$

20. $\frac{x}{4.1} = 3.9.$

21. $\frac{x}{5.2} = 1.7.$

22. $\frac{x}{0.61} = 0.75.$

23. What number divided by 26 equals 26?

24. What number divided by $3\frac{1}{2}$ equals $4\frac{3}{4}$?

25. If $\frac{1}{8}$ of a certain distance is 29.4 ft., what is the distance?

26. If $\frac{1}{32}$ of the thickness of a steel plate is $\frac{1}{8}$ in., what is the thickness of the plate?

27. If $\frac{1}{48}$ of the diameter of a steel rod is $\frac{1}{96}$ in., what is the diameter of the rod?

28. If 1% of the cost of a house is \$47.50, what is the cost of the house?

18. Axioms. A general statement admitted to be true without proof is called an *axiom*.

The axioms needed in the beginning of algebra are six in number. Others will be added as required.

1. *If equals are added to equals, the results are equal.*
2. *If equals are subtracted from equals, the results are equal.*
3. *If equals are multiplied by equals, the results are equal.*
4. *If equals are divided by equals, the results are equal.*
5. *Like powers or like roots of equals are equal.*
6. *Quantities equal to the same quantity are equal to each other.*

Exercise 15. Use of the Axioms

Examples 1 to 7, oral — Examples 8 to 19, written

1. If $2a = 16$, what does a equal? What axiom is used?

2. If $2a + 1 = 9$, what does $2a$ equal? What axiom is used? Then what does a equal? What axiom is used?

3. If $3a - 1 = 14$, what does $3a$ equal? Then what does a equal? What two axioms are used?

4. If $\frac{1}{3}a + 1 = 7$, what does $\frac{1}{3}a$ equal? Then what does a equal? What two axioms are used?

5. If $\frac{1}{5}a - 1 = 4$, what does $\frac{1}{5}a$ equal? Then what does a equal? What two axioms are used?

6. If $\frac{2}{3}a = 12$, what does $\frac{1}{3}a$ equal? Then what does a equal? What two axioms are used?

7. If we divide both members of the equation $\frac{2}{3}a = 12$ by $\frac{2}{3}$, what is the result? What axiom is used?

Find the value of a in the following:

8. $3a = 111$.
9. $7a = 112$.
10. $8a = 112$.
11. $9a = 135$.
12. $3a + 1 = 112$.
13. $7a + 6 = 118$.
14. $8a + 9 = 121$.
15. $9a - 5 = 130$.
16. $\frac{2}{3}a = 70$.
17. $\frac{3}{4}a = 66$.
18. $\frac{1}{5}a = 72$.
19. $\frac{5}{8}a = 75$.

19. Algebra and Arithmetic compared. If to twice a certain number we add 7, the result is 33. Required the number.

Solution by Algebra

Let n stand for the number.

Then $2n$ is twice the number,

and $2n + 7 = 33$, as stated in the problem.

Therefore $2n = 26$, by subtracting 7 from equals,

and $n = 13$, by dividing equals by 2.

Check or *Proof.* $2 \times 13 + 7 = 33$. Therefore the work is correct.

Solution by Arithmetic

Because twice the number added to 7 equals 33, therefore if 7 is taken away from 33 there will remain twice the number. Therefore 26 is twice the number. Therefore once the number is half of 26, or 13.

$$\begin{array}{r} 33 \\ 7 \\ \hline 2\,)\,26 \\ \hline 13 \end{array}$$

20. How to solve a Problem. From the above problem we see that the algebraic solution makes the reasoning clearer. To solve a problem by algebra,

(1) *Write a letter for the number sought;*

(2) *Use this letter in the statement of the problem;*

(3) *This will give an equation representing the problem;*

(4) *Find the value of the letter; that is, solve the equation.*

Exercise 16. Solution of Problems

Examples 1 to 6, oral — Examples 7 to 21, written

1. If we wish to find a certain number such that if 7 is added to 3 times this number the result is 40, by what shall we represent this number in an equation?

2. Then how shall we represent 3 times the number?

3. How shall we represent this added to 7?

4. How shall we express this sum as equal to 40?

5. What shall we do to these equals so as to leave n alone on one side of the equation?

6. How shall we check or prove that our result is correct?

7. If to a certain number we add 11, the result is 29. What is the number?

8. If to 7 times a certain number we add 3, the result is 45. What is the number?

9. If to 23 times a certain number we add 47, the result is 300. What is the number?

10. If from 19 times a certain number we subtract 17, the result is 173. What is the number?

11. If from 2.3 times a certain number we subtract 2.8, the result is 13.3. What is the number?

12. If from $3\frac{4}{5}$ times a certain number we subtract $2\frac{1}{5}$, the result is $16\frac{4}{5}$. What is the number?

13. If to twice a certain number we add $3\frac{7}{8}$, the result is $4\frac{1}{8}$. What is the number?

14. If from twice a certain number we subtract $3\frac{7}{8}$, the result is $4\frac{1}{8}$. What is the number?

15. The length of a room exceeds the width by 6 ft. The width is 18 ft. What is the length? What is the area of the floor?

16. Twice the width of a room is 8 ft. more than the length of the room. The width is 12 ft. What is the length?

17. If to 4 times the length of a room we add 6 ft., the result is 70 ft.; and if to 5 times the width of the room we add 10 ft., the result is also 70 ft. Find the length and width.

18. A biplane traveling at the rate of d miles a minute goes 8.4 mi. in 7 min. Find the value of d.

19. A train traveling at the rate of m miles an hour goes $112\frac{1}{2}$ mi. in $2\frac{1}{2}$ hr. Find the value of m.

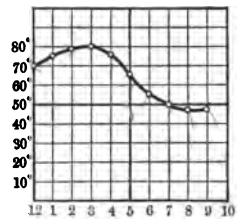
20. If a man's annual income is increased by \$600, the result is the same as twice his annual income decreased by \$1800. What is his annual income?

21. If a man's annual income is increased by \$100, the result is the same as three times his annual income decreased by \$3900. What is his annual income?

CHAPTER II

NEGATIVE NUMBERS

21. Curve Tracing. In this figure the successive hours of the day from noon to 9 P.M. are represented on a horizontal line, and the temperatures are represented by points on the vertical lines. It shows that at noon the temperature was 70°, at 1 P.M. 75°, at 2 P.M. 78°, at 3 P.M. 80°, and so on. If we connect these points by a curve, this line gives us a picture of the change in temperature for nine hours.



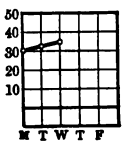
Exercise 17. Curve Tracing

Examples 1 to 5, oral — Examples 6 to 12, written

1. By the above curve, what was the temperature at 4 P.M.?
2. At what time did the temperature stop rising?
3. What was the temperature at 5 P.M.? at 6 P.M.?
4. At what time did the temperature fall below 50°?
5. Did the temperature reach 45° before 9 P.M.?

Rule some paper and draw lines to show the following:

6. The attendance in a certain algebra class was 30 on Monday, 32 on Tuesday, 34 on Wednesday, 30 on Thursday, and 28 on Friday. Part of the line (in this case made up of short straight lines) is here shown. Trace the entire line. Make a similar problem and trace the line.



Rule some paper and draw lines to show the following:

7. The season's record of a certain baseball team is as follows: To May 1 the per cent of games won to games played was 68.4%; to June 1 it was 67.6%; to July 1, 67.2%; to Aug. 1, 63.7%; to Sept. 1, 64.2%; to Oct. 1, 64.7%. The line is here shown. Draw a similar line for these per cents: 65%, 66%, 65.5%, 64.2%, 63.5%, 62%. Draw a similar line for these per cents: 54%, 48%, 39%, 45%, 52%, 55.5%.

8. The pressure on the steam gauge of a boiler varies as follows: 6 A.M., 120 lb.; 7 A.M., 122 lb.; 8 A.M., 132 lb.; 9 A.M., 130 lb.; 10 A.M., 126 lb.; 11 A.M., 130 lb.; noon, 124 lb.; 12.30 P.M., 120 lb.; 1 P.M., 125 lb.; 2 P.M., 130 lb.; 3 P.M., 128 lb.; 4 P.M., 126 lb.; 5 P.M., 124 lb.; 6 P.M., 112 lb.

9. A boy's height from the age of 5 to the age of 15, stated in inches, varied as follows: 5 yr., 42; 6 yr., 44; 7 yr., 46; 8 yr., 49; 9 yr., 52; 10 yr., 54; 11 yr., 56; 12 yr., 58; 13 yr., 61; 14 yr., 63; 15 yr., 68.

10. A girl's height from the age of 5 to the age of 15, stated in inches, varied as follows: 5 yr., 42; 6 yr., 44; 7 yr., 45; 8 yr., 48; 9 yr., 50; 10 yr., 52; 11 yr., 54; 12 yr., 58; 13 yr., 60; 14 yr., 63; 15 yr., 65.

11. The population of the United States, in millions, for various years was as follows: 1820, 10; 1830, 13; 1840, 17; 1850, 23; 1860, 31; 1870, 39; 1880, 50; 1890, 63; 1900, 76; 1910, 92. In such a case it is convenient to use 0.1 in. to represent 10 millions.

12. In order to determine the course of a tunnel a cross-section plan of a river bed had to be made from soundings taken at various distances from the left bank, the measurements being made in feet. Draw a plan of the river bed from the following data:

Distance from bank,	0, 10, 20, 30, 40, 45, 50, 51, 52, 55, 60.
Depth of river,	9, 12, 15, 25, 25, 28, 35, 37, 35, 20, 8.

22. Numbers below Zero. When it is necessary to distinguish between temperature below zero and temperature above zero, we write 10° above zero, + 10°; and 10° below zero, − 10°.

If the temperature is 20° above zero and it decreases 15°, it is then 5° above zero, or + 5°. If it decreases 5° more, it is then 0°. If it decreases 5° more, it is 5° below zero, or − 5°. If it decreases 20° more, it is 25° below zero, or − 25°.

We therefore find a new meaning for the signs + and −. They not only indicate addition and subtraction (signs of *operation*), but they tell on which side of zero a number is (sign of *quality*).

23. Positive Numbers. The ordinary numbers which we use in arithmetic are called *positive numbers.*

Thus 3°, 3 in., $\frac{3}{4}$, $\sqrt{3}$, are all positive numbers. If we wish to make this fact emphatic, we may write them thus: + 3°, + 3 in., + $\frac{3}{4}$, + $\sqrt{3}$, but otherwise the + sign is unnecessary here. The expression + 3 is read "positive 3" or "plus 3."

24. Negative Numbers. Numbers on the other side of zero from positive numbers are called *negative numbers.*

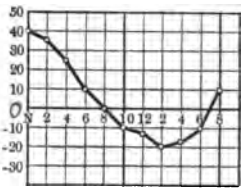
Thus − 3° is a negative number. If distance upwards, above the earth's surface, is called positive, distance below the surface is called negative, so that we may have + 10 ft. and − 10 ft. The expression − 3 is read "negative 3" or "minus 3." We may think of zero as either positive or negative, since it divides the two classes of numbers.

If the temperature is 20° below zero, or − 20°, and it increases 7°, it is then 13° below zero, or − 13°. If it increases 9° more, it is 4° below zero, or − 4°. If it increases 4° more, it is then zero, or 0°. If it increases 32° more, it is then 32° above zero, or + 32°.

25. Absolute Value. The numerical value of a quantity, without reference to its sign, is called its *absolute value.*

The absolute value of − 4 is 4, and that of $-a$ is a.

26. Curve Tracing with Negative Numbers. If the temperature in St. Paul, on a winter's day, falls from + 40° at noon to − 20° at 2 A.M., and then rises again to + 10° at 8 A.M., as here shown, the curve tells us that it was below zero from 8 P.M. until about 7 A.M.



In this case we represent negative numbers below the horizontal line, which represents zero.

Exercise 18. Curve Tracing

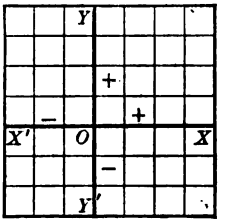
Examples 1 to 7, oral — Examples 8 to 10, written

1. By the above curve, when was the temperature 35°?
2. At what times was it − 10°? + 10°? + 5°? − 5°?
3. What was the temperature at 6 A.M.? at 8 A.M.?
4. At what time did the temperature cease falling?
5. How much did it rise from 2 A.M. to 6 A.M.? to 8 A.M.?
6. What was the difference in temperature at 6 P.M. and 10 P.M.? at 8 P.M. and 2 A.M.? at 10 P.M. and 2 A.M.?
7. When the temperature is − 10°, how much must it rise to be 0°? to be + 10°? to be + 20°?

Trace the curves to show the following variations in temperature for the twenty-four hours:

8. Noon, 33°; 2 P.M., 42°; 4 P.M., 34°; 6 P.M., 30°; 8 P.M., 22°; 10 P.M., − 8°; midnight, − 8°; 2 A.M., − 10°; 4 A.M., − 5°; 6 A.M., 0°; 8 A.M., 12°; 10 A.M., 30°; noon, 34°.
9. Noon, 48°; 2 P.M., 45°; 4 P.M., 40°; 6 P.M., 25°; 8 P.M., 0°; 10 P.M., − 5°; midnight, − 8°; 2 A.M., − 5°; 4 A.M., 0°; 6 A.M., 5°; 8 A.M., 10°; 10 A.M., 20°; noon, 35°.
10. Noon, 0°; 2 P.M., 5°; 4 P.M., 10°; 6 P.M., 8°; 8 P.M., 0°; 10 P.M., − 5°; midnight, − 10°; 2 A.M., − 5°; 4 A.M., − 2°; 6 A.M., 10°; 8 A.M., 20°; 10 A.M., 32°; noon, 45°.

27. Other Uses of Negative Numbers. If we call some special point on a line zero (0), we usually call distances to the right positive and distances to the left negative, just as we call distances up (as on the thermometer) positive and distances down negative. But because we usually call weight positive, we speak of the weight of a balloon (which pulls upwards) negative.



The following are some additional illustrations of negative numbers:

If a man is worth \$1000, we may say that he has + \$1000; but if he is \$1000 in debt, we may say that he is worth − \$1000.

If we have 25 on a score in a game, we have + 25; but if we are 25 worse off than nothing, we have − 25.

If we call latitude north of the equator positive, we may call south latitude negative.

If we call longitude west of Greenwich positive, we may call east longitude negative; and if we call west longitude negative, we should call east longitude positive.

If we call the motion of a piston rod of an engine positive when it is to the right, we may call it negative when it is to the left.

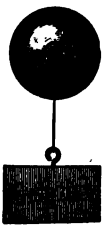
If we call downward pressure positive, we may speak of upward pressure as negative.

If we call distance above the earth's surface positive, we may call distance below the earth's surface negative.

We therefore see that negative numbers are just as real as positive numbers, for the temperature is just as real when the thermometer indicates that it is below zero as it is when the mercury rises above zero, and a man's debts are just as real as his capital.

In ancient times people used only whole numbers (integers). Other kinds of numbers were invented as they became necessary, and these are sometimes called artificial numbers. Artificial numbers like $\frac{3}{4}$, $\sqrt{3}$, and − 3 all have their uses, as we have seen, not only in the theory of algebra but in its practical applications.

28. Adding Negative Numbers. If we tie to a 10-pound weight a toy balloon that pulls upward 1 pound, what will the two together weigh? From the answer to this question we find the following:



To add a positive number to a negative number, take the difference of their absolute values and prefix the sign of the numerically greater number.

Thus + 10 lb. and − 1 lb. are + 9 lb.;
+ 10 lb. and − 10 lb. are 0 lb.;
+ 10 lb. and − 15 lb. are − 5 lb.

Similarly, *to add a negative number to a negative number, take the sum of their absolute values and prefix the negative sign.*

Illustrate this, using two balloons, one pulling upward 5 lb. and the other pulling upward 6 lb.

Exercise 19. Addition

Examples 1 to 5, oral — Examples 6 to 36, written

1. What is the combined weight of + 25 lb. and − 5 lb.?

2. What is the combined weight of 30 lb. and − 60 lb.?

3. A freight engine is switching in front of a station. If it runs 500 ft. to the right of the station (+ 500 ft.) and then backs 525 ft. (− 525 ft.), how many feet is it from the station? (Add 500 and − 525.)

4. In drilling a well the drill is raised + 8 ft. above the surface of the ground. It is then dropped 16 ft. (− 16 ft.). Where is it then with respect to the surface? (Add 8 and − 16. A negative distance above the surface means below the surface.)

5. A boy is fishing in deep water with a line 22 ft. long. If the tip of the pole is + 6 ft. above the water, how far is the sinker from the surface of the water if it is 3 ft. from the hook? (Add 6, − 22, and 3.)

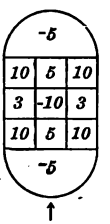
Add as indicated:

6. $-275+316$.	**9.** $-481+296$.	**12.** $243+(-48)$.
7. $-486+531$.	**10.** $-370+198$.	**13.** $300+(-97)$.
8. $-279+603$.	**11.** $-760+436$.	**14.** $421+(-84)$.

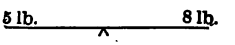
15. A man who was worth \$4500 lost \$1750 and then earned \$900. How much was he then worth? (Add 4500, -1750, and 900.)

16. A man who was \$450 in debt contracted another debt of \$250. He then earned \$1000. How much was he then worth?

17. A game is played by throwing bean bags in the direction of the arrow. Suppose the score stands -5, 5, 3, 10, -10, 5, 10, 10, 3, 3, -5, how much is the total score?



18. If this board without any weights at the ends just balances, and if I put 5 lb. at one end and 8 lb. at the other end, how much must I add to the 5 lb. to make it balance? Instead of adding to the 5 lb., how much must I add to the 8 lb.?



19. A boat that goes 14.1 mi. an hour in still water is going against a stream flowing 3.8 mi. an hour. What is the rate at which the boat will travel? (14.1 mi. and -3.8 mi. are how many miles?)

20. If a mine is opened 400 ft. above the base of a mountain and a shaft is sunk 750 ft., how much is the base of the shaft above or below the base of the mountain?

Add the following:

21.	**22.**	**23.**	**24.**	**25.**	**26.**
-62	75	60	-30	-90	-35
62	-80	-60	-40	-72	-75

27. Find the average noon temperature for the week in which the noon temperatures were 15°, 3°, 0°, $-7°$, $-20°$, 6°, 25°.

28. A mass of iron and wood is placed in a tank of water. The iron tends to sink the mass with a force of 20 lb., and the wood tends to buoy it up with a force of 16 lb. What does the mass weigh under water?

29. If my watch is 5 min. faster than the schoolroom clock, and the clock is 7 min. slower than the correct time, how near is my watch to the correct time?

30. In a tug of war one group of boys pulls to the north with a force of $256\frac{3}{4}$ lb., and the other group pulls to the south with a force of $252\frac{1}{2}$ lb. What is the resulting force?

31. On a rock that is 6 ft. 8 in. below the average level of the sea a lighthouse is built 81 ft. 4 in. high. How high is the top of the lighthouse above the average sea level?

32. At a point on a hill 329 ft. above a valley the shaft of a mine is sunk to a depth of 401 ft. How far is the bottom of the shaft below the valley?

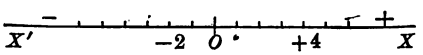
33. An aëroplane that can fly 58.2 mi. an hour in still air is flying against a wind that retards it 9.7 mi. an hour. At what rate does the aëroplane fly?

34. An elevator starting from the main floor of a high building runs up 9 stories, then down 10 stories, and then up 1 story. Where is it then? Express the solution by using positive and negative numbers.

35. A certain office building is 37 stories above the street and 4 stories below. An elevator starting at the street level ascends 28 stories, descends 30, ascends 35, descends 37, and ascends 26. Where is the elevator then? Represent the movements by a diagram, using positive and negative signs.

36. A man having \$374.75 in the bank deposits \$176.50 on Monday, checks out \$482.60 on Tuesday, deposits \$243.85 on Wednesday, and checks out \$281.45 on Thursday. What is his balance? Add the positive numbers, then the negative numbers, and then the two sums.

29. Subtracting a Negative Number. If the temperature is $-10°$ at midnight and $+40°$ at noon, the difference in temperature is evidently 50°, for the mercury must rise 10° to reach 0°, and 40° more to reach $+40°$. Likewise,



in this figure the difference between -2 and $+4$ is 6; for a point must move 2 spaces to get from -2 to 0, and 4 more to reach $+4$; that is, 6 must be added to -2 to make 4.

To subtract a negative number we may obtain the same result by adding a positive number with the same absolute value.

That is, $4-(-2)=4+2=6$. Likewise $-5-(-3)=-5+3=-2$. This is the usual rule for subtracting a negative number, but it is never necessary actually to change the sign of the subtrahend. If we subtract a smaller from a larger number, such as -10 from -4, the result is necessarily positive. If we subtract a larger from a smaller number, the result is necessarily negative.

Exercise 20. Subtraction

Examples 1 to 8, oral — Examples 9 to 16, written

1. How much difference in price is there in selling a horse at \$25 below cost or at \$30 above cost?

2. The temperature on one morning was $+11°$, and the next morning $-7°$. What was the difference in temperature?

3. If there is a house for every number, how many houses would you pass in going from 48 East Washington Street to 17 West Washington Street, including both these houses?

4. Jefferson Street is 6 blocks east of Adams Street, and Monroe Street is 9 blocks west of Adams Street. Monroe Street is how many blocks west of Jefferson Street?

5. $4-(-3)$.
6. $5-(-7)$.
7. $6-(-3)$.
8. $7-(-7)$.
9. $0-(-47)$.
10. $36-(-49)$.
11. $59-(-59)$.
12. $78-(-96)$.
13. $13.7-(-27.8)$.
14. $16.5-(-43.4)$.
15. $37.8-(-96.8)$.
16. $52.7-(-88.9)$.

30. Subtracting a Positive Number. If the temperature is $-10°$ at midnight and falls $2°$ more during the next hour, it is then $-12°$. That is, to subtract $2°$ from $-10°$ is the same as to add $-2°$ to $-10°$. Therefore

In subtracting a positive number we may obtain the same result by adding a negative number with the same absolute value.

Hence $(+a)-(+b)=(+a)+(-b)=a-b$. It is not necessary to remember such a rule, for if we are taking a smaller from a larger number, the result is positive, and in the contrary case it is negative.

Exercise 2.. Subtraction

Examples 1 to 17, oral — Examples 18 to 29, written

1. How much is $10°-5°$? $0°-5°$? $-5°-5°$?
2. How much is \$20 − \$10? \$0 − \$10? \$5 − \$10?
3. How much is 15 ft. − 6 ft.? 0 ft. − 6 ft.? −12 ft. − 6 ft.?
4. How much is $12¢-7¢$? $7¢-7¢$? $5¢-7¢$?
5. How much is 25 lb. − 15 lb.? 15 lb. − 15 lb.? 5 lb. − 15 lb.?

6. $17-7$.	**9.** $6-5$.	**12.** $7-3$.	**15.** $-4-7$.
7. $17-17$.	**10.** $4-5$.	**13.** $1-3$.	**16.** $-5-6$.
8. $17-27$.	**11.** $-4-5$.	**14.** $-1-3$.	**17.** $-8-3$.

18. If a man has \$178 and incurs a debt of \$275, how much is he then worth?

19. If a man has − \$178 (that is, if he is \$178 in debt) and incurs a debt of \$275, how much is he then in debt?

20. $69-32$.	**23.** $48-75$.	**26.** $-36-92$.
21. $69-72$.	**24.** $52-86$.	**27.** $-49-76$.
22. $-69-32$.	**25.** $73-91$.	**28.** $-56-83$.

29. If two trains pass each other in the opposite direction at 9.45 A.M., one going at the rate of 47.6 mi. an hour, and the other at the rate of 39.7 mi. an hour, how far apart will they be at 10.15 A.M. if these rates are maintained?

31. Multiplying and Dividing Negative Numbers. We multiply and divide negative numbers just as we multiply and divide positive numbers. If a man has − \$5 (is \$5 in debt), he will have − \$10 if he is twice as much in debt.

That is, $2 \times (-\$5) = -\$10$,

and $-\$10 \div 2 = -\$5$.

Exercise 22. Multiplication and Division

Examples 1 to 4, oral — Examples 5 to 20, written

1. If one balloon pulls up 400 lb. (weighs − 400 lb.), what will be the upward pull of 3 such balloons? Represent the weight as a negative number.

2. If the thermometer indicates − 8°, what is the temperature when it indicates half as much below zero? When it indicates twice as much below zero?

3. If a carrier pigeon can fly 38 mi. an hour in still air, at what rate will it fly against a 15-mile wind? against a wind that blows twice as fast? against a hurricane that blows three times as fast?

4. A checker of bales of cotton finds one bale 16 lb. short, a second bale twice as much short, and a third bale half as much short in weight. Express these shortages of weight in algebraic language.

5. A man's debts amounted last year to \$475. The year before they were 4 times as much. This year he has paid his debts and has \$825 in the bank. What is the difference between his financial standing year before last and now?

6. $3 \times (-86)$.	**11.** $1.7 \times (-3.2)$.	**16.** $-43.4 \div 7$.
7. $4 \times (-49)$.	**12.** $2.8 \times (-4.9)$.	**17.** $-5.58 \div 9$.
8. $6 \times (-73)$.	**13.** $-125 \div 25$.	**18.** $-7.92 \div 9$.
9. $7 \times (-89)$.	**14.** $-493 \div 17$.	**19.** $-40.7 \div 3.7$.
10. $8 \times (-96)$.	**15.** $-4.93 \div 1.7$.	**20.** $-5.39 \div 4.9$.

32. Multiplying by a Negative Number. We cannot pick up a book $2\frac{2}{3}$ times. Nevertheless we say that $2\frac{2}{3}$ times \$3 equals \$8. That is, we define what is meant by multiplying by $2\frac{2}{3}$, and then we use the word "times" just as we do with integers.

Similarly, we cannot pick up a book -2 times, but we may define what we mean by multiplying by -2, and then we may use the word "times" as we do with positive integers.

Because $3 \times (-2) = -6$, therefore -2×3 ought to equal -6. Therefore we define multiplication by a negative number to mean multiplication by a positive number having the same absolute value, the sign of the product being then changed.

Therefore

$$2 \times (-3) = -6, \qquad a \cdot (-b) = -ab,$$
$$-2 \times 3 = -6, \qquad -a \cdot b = -ab,$$
$$-2 \times (-3) = 6, \qquad -a \cdot (-b) = ab.$$

If two numbers have like signs, their product is positive; if they have unlike signs, their product is negative.

Exercise 23. Multiplication

Examples 1 to 7, oral — Examples 8 to 13, written

1. If each of 3 men spends \$2 in a store, how much does the store receive? How much is $3 \times \$2$?

2. If each of 3 men steals \$2 from a store, how much does the store gain or lose? How much is $3 \times (-\$2)$?

3. If 3 men who would have spent \$2 each in a store are persuaded to trade elsewhere, how much does the store gain or lose in gross receipts? How much is $-3 \times \$2$?

4. If 3 men who would have stolen \$2 each from a store are arrested before the theft, how much does the store gain?

5. $3 \times (-9)$.
6. $-3 \cdot 9$.
7. $-3 \cdot (-9)$.
8. $-27 \cdot 63$.
9. $-42 \cdot (-76)$.
10. $-29 \cdot (-38)$.
11. $-96 \cdot 83$.
12. $-2 \cdot (-3) \cdot (-7)$.
13. $-8 \cdot (-8) \cdot (-9)$.

33. Dividing by a Negative Number. Division being the inverse of multiplication, because

$$2 \times 3 = 6, \text{ therefore } 6 \div 3 = 2;$$
$$2 \times (-3) = -6, \text{ therefore } -6 \div (-3) = 2;$$
$$-2 \times 3 = -6, \text{ therefore } -6 \div 3 = -2;$$
$$-2 \times (-3) = 6, \text{ therefore } 6 \div (-3) = -2.$$

If two numbers have like signs, their quotient is positive; if they have unlike signs, their quotient is negative.

Exercise 24. Division

Examples 1 to 16, oral — Examples 17 to 33, written

1. How much is $2 \cdot (-7)$? $-14 \div 2$? $-14 \div (-7)$?
2. How much is $-3 \cdot 8$? $-24 \div (-3)$? $-24 \div 8$?
3. How much is $-7 \cdot (-9)$? $63 \div (-7)$? $63 \div (-9)$?
4. How much is $-a \cdot (-b)$? $ab \div (-a)$? $ab \div (-b)$?

5. $25 \div (-5)$.
6. $-25 \div 5$.
7. $-25 \div (-5)$.
8. $-25 \div (-25)$.
9. $36 \div (-4)$.
10. $-36 \div 4$.
11. $-36 \div (-4)$.
12. $-36 \div (-9)$.
13. $56 \div (-8)$.
14. $56 \div (-7)$.
15. $-56 \div (-8)$.
16. $-56 \div (-7)$.

17. By what number must 17 be multiplied to make 544? to make -544?
18. By what number must -19 be multiplied to make -399?
19. By what number must -22 be multiplied to make 770? to make -770?

20. $625 \div 25$.
21. $625 \div (-25)$.
22. $-625 \div 25$.
23. $-625 \div (-25)$.
24. $-675 \div (-25)$.
25. $3367 \div 37$.
26. $3367 \div (-37)$.
27. $-3367 \div (-37)$.
28. $-3367 \div (-91)$.
29. $34.3 \div 7$.
30. $34.3 \div (-7)$.
31. $-34.3 \div 7$.
32. $-34.3 \div (-7)$.
33. $-34.3 \div (-0.7)$.

34. The System of Integers. We now see that our system of integers extends indefinitely on both sides of zero, thus:

$$\cdots -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6 \cdots$$

35. The Properties of Zero. The following are the important properties of zero:

$$a + 0 = a, \qquad 0 - a = -a, \qquad 0 \cdot a = 0,$$
$$a - 0 = a, \qquad a \cdot 0 = 0, \qquad 0 \div a = 0.$$

The expression $a \div 0$ may be thought of for the present as having no meaning, division by zero not being allowed. It is considered later.

Exercise 25. Review

Examples 1 to 32, oral — Examples 33 to 44, written

1. $4 + (-3)$.
2. $6 + (-9)$.
3. $9 + (-3)$.
4. $1 + (-9)$.
5. $7 + (-3)$.
6. $-2 + 8 \times 2 \div 4$.
7. $-8 + 4 \div 2 \times 3$.
8. $-5 + 7 \times 3$.
9. $-5 - 9 \div 9$.
10. $-7 - 2 \times 4$.
11. $2 \times (-7) \times 3$.
12. $5 \times (-3) - 2$.
13. $8 \times (-6) + 6$.
14. $-5 \times 7 - 4$.
15. -8×9.
16. $-7 \times (-8)$.
17. $8 \div (-2)$.
18. $-8 \div 2$.
19. $-6 \div 2$.
20. $-8 \div (-2)$.
21. $-7 \div (-1)$.

22. How much is $-3 \times (-3)$? $(-3)^2$? $(-7)^2$? $(-9)^2$?

23. How much is $9 - 2 \times (-2) \times (-2)$? $9 - (-2)^3$?

24. $(-1)^2$. **26.** $(-2)^4$. **28.** $(-5)^2$. **30.** $(-1)^6$.
25. $(-1)^5$. **27.** $(-2)^5$. **29.** $(-6)^2$. **31.** $(-1)^7$.

32. If $x = -2$, what is the value of x^2? of $-x$? of $(-x)^2$?

If $a = 4$ and $b = -3$, find the value of:

33. $a + b$. **35.** $-b$. **37.** $(-b)^2$. **39.** $-3ab$.
34. $a^2 - 2b$. **36.** $a^2 - \frac{1}{3}b$. **38.** $3a^2b^2$. **40.** $a^2 - b^2$.

If $a = 5$ and $b = -3$, find the value of:

41. $(a + b)(a - b)$.
42. $a^2 + 2ab + b^2$.
43. $a^3 + 3a^2 + b$.
44. $a^3 + 3a^2b + 3ab^2 + b^3$.

CHAPTER III

ALGEBRAIC EXPRESSIONS

36. Terms used in Algebra and Arithmetic. As already seen, many of the terms of mathematics are used in algebra exactly as in arithmetic. Thus we speak of addition, addends, and sum; of subtraction, minuend, subtrahend, and remainder; of multiplication and division, with the various numbers entering into these processes; of fraction, numerator, and denominator; and of various other operations and terms, these being used in algebra in the same way that they are used in arithmetic. Since they are well known, such terms usually do not require further definition, although a few are formally defined at this time for future reference.

It should always be remembered that the letters of algebra represent numbers. This is the reason why the terms used in arithmetic may properly be employed in connection with algebraic expressions.

37. Factor. Any one of two or more numbers which multiplied together form a product is called a *factor* of the product.

Thus just as 2 and 3 are factors of 6, so a and b are factors of ab; $(b + b')$ and h are factors of $(b + b')h$; m and m are factors of m^2; and 2, π, and r are factors of $2\pi r$.

38. Literal and Numerical Factors. A factor containing a letter is called a *literal* factor; one that is expressed by a numeral is called a *numerical* factor.

In most cases factors are considered to be integers. In an expression like $2\pi r$, for example, π is considered a factor because it has the form of an integer, although it equals $3.1416-$; and r is considered a factor, although its numerical value may be fractional. Similarly, we may speak of $\sqrt{5}$ as a factor of $2\sqrt{5}$, since no misunderstanding will arise from so doing. We would not, however, say that 5 is factorable, although it equals $\sqrt{5} \cdot \sqrt{5}$.

39. Coefficient. If an expression is the product of two factors, either factor is called the *coefficient* of the other.

Thus in the expression ab, a is the coefficient of b, and b is the coefficient of a.

The factor that is considered the coefficient is usually written first. Thus in $2\pi r$, 2 is the coefficient of πr, and 2π is the coefficient of r.

The coefficient 1 is omitted, x being the same as $1x$.

40. Power. The product of several equal factors is called a *power*.

Thus $2 \times 2 \times 2 = 2^3$, or 8; and 2^3, or 8, is the third power of 2. Similarly, as already stated, aaa is the third power of a, and is written a^3.

Other kinds of powers will be considered later.

41. Exponent. The number placed to the right and slightly above another to indicate a power is called an *exponent*.

Thus in a^3, 3 is the exponent of a.

In $2r^3$, 2 is the coefficient of r^3, and 3 is the exponent of r. The two should be carefully distinguished. *The coefficient shows the number of equal addends; the exponent shows the number of equal factors.*

A letter without an exponent is considered as having the exponent 1.

42. Root. One of several equal factors of an expression is called a *root*.

For example, we have square roots and cube roots, as already defined in § 7. Furthermore, since $2^4 = 16$, it follows that 2 is the fourth root of 16. That is, $\sqrt[4]{16} = 2$.

43. Absolute Term. If a polynomial contains a numerical term, this is called the *absolute term*.

Thus in the polynomial $a^2 + 3a - 4$ the absolute term is -4, and in the polynomial $3x^4 + \sqrt{2}$ the absolute term is $\sqrt{2}$.

44. Similar Terms. Monomials that have a common factor are called *similar terms* or *similar monomials*.

Thus a, $3a$, and $-7a$ are similar with respect to a; $2\sqrt{5}$ and $-\frac{2}{5}\sqrt{5}$ are similar with respect to $\sqrt{5}$; $2 \cdot (-7)$ and $4 \cdot (-7)$ are similar with respect to -7; and ax^2 and bx^2 are similar with respect to x^2.

Terms that are not similar are said to be *dissimilar*.

Exercise 26. Algebraic Expressions

Examples 1 to 31, oral — Examples 32 to 45, written

1. What are the factors of ab? of a^2? of a^2b? of $3a^2b$?
2. What are the terms of $a+b$? of $a+2b$? of $a^2-2ab+b^2$?
3. What are the numerical coefficients in a? $3a$? $-\frac{3}{4}ab^3$?
4. What are the exponents in a^2? $3a^4$? a^2b^3? $x^2y^3z^4$?
5. What other way is there of writing $a+a+a$? $a \cdot a \cdot a$?
6. What monomials form the trinomial $a^2-2ab+b^2$?
7. From $2a^2, -7a^3, -6a, -\frac{1}{2}a^2, -5a, 4a^3, 7a^2, -3a^2, 2a^3$, select four similar terms; three other similar terms.

If $a=2$, find the value of:

8. a^2.	11. $2a$.	14. $\frac{1}{2}a$.	17. $-7a$.	20. $4a^2$.	23. $\frac{1}{2}a^3$.
9. a^3.	12. $3a$.	15. $\frac{3}{2}a$.	18. $-9a$.	21. $2a^3$.	24. $\frac{1}{4}a^4$.
10. a^4.	13. $4a$.	16. $\frac{5}{2}a$.	19. $-6a$.	22. $8a^4$.	25. $\frac{1}{8}a^5$.

If $a=3$ and $b=-2$, find the value of:

26. $a+b$.	28. a^2-b.	30. $2a+b$.
27. $a-b$.	29. a^2+b^2.	31. $2a-b$.

If $x=2$ and $y=-3$, find the value of:

32. $5x^2+7y^2$.	34. x^3+y^3.	36. x^4+y^4.
33. $7x^3-2y^2$.	35. $x^2+2xy+y^2$.	37. x^4+3y^4.

38. The factors of a monomial are a, x, b, a, 4, x, and y. Write the monomial in the usual way.

39. The terms of a polynomial are $3x^2y$, x^3, y^3, and $3xy^2$. Write the polynomial, beginning with the highest power of x and letting the exponents of x decrease by 1 to the right.

If $x=10$, find the value of:

40. $2x+1$.	43. $4x^3+2x^2+3x+7$.
41. $3x^2+2x+1$.	44. $5x^3+5x^2+5x+5$.
42. $5x^2+9x+1$.	45. $x^4+x^3+x^2+x+1$.

45. Function. An algebraic expression that depends upon another for its value is called a *function* of the latter.

For example, the formula for the circumference of a circle is $c = 2\pi r$. Here the factors 2 and π are definite numbers, but r may have any value, and the value of c depends upon the value we give to r. Therefore c is called a *function* of r.

Similarly, in the area of a circle we have $a = \pi r^2$. Here a depends upon the value of r, and therefore is a function of r.

It is customary to write $f(x)$ or $F(x)$ for "function of x." The symbol $f(x)$ is also read simply "f of x." $F(x)$ may be read "f major of x."

The function idea is very important, and teachers are advised to call attention to it in all formulas. It is considered more fully in Book II.

Exercise 27. Algebraic Expressions

Examples 1 to 4, oral — Examples 5 to 12, written

1. Give two illustrations of a function of r.

2. If $f(x) = x^2 + x + 1$, what is $f(2)$; that is, what does $x^2 + x + 1$ equal when 2 is substituted for x?

3. If $f(r) = 2\pi r$, what is $f(7)$? Take $\frac{22}{7}$ for π.

4. If $f(r) = \pi r^2$, what is $f(1)$? Take $3\frac{1}{7}$ for π.

5. If $f(a) = a^2 + 2a + 1$, what is $f(7)$? $f(7.5)$?

6. Write in briefer form $m + m + m + m + m$.

7. Write in briefer form $mmmmm$.

8. By factoring the numbers, find: $\sqrt{25}$, $\sqrt[3]{27}$, $\sqrt[5]{243}$.

9. If it is known that a rectangle is 20 in. long and h inches high, the area is a function of what quantity? What is the area when $h = 2$? 4? 10?

10. If it is known that a triangle is 10 in. high and has a base b inches, the area is a function of what quantity? What is the area when $b = 20$? 30? 40? 50?

11. In the formula $c = \pi d$, c is a function of what letter? Give the various values of this function for $d = 0, 1, 2, 3, 4, 5$, using $3\frac{1}{7}$ for the value of π in this and similar cases.

12. If $f(r) = \frac{4}{3}\pi r^3$, find $f(1)$, $f(2)$, $f(3)$, and $f(4)$.

CHAPTER IV

ADDITION

Exercise 28. Addition of Similar Monomials

Examples 1 to 22, oral — Examples 23 to 31, written

1. Add 2 ft. and 3 ft.; $2f$ and $3f$; $2 \cdot 5$ and $3 \cdot 5$.
2. Add $7¢$ and $9¢$; $7c$ and $9c$; $7 \cdot 2$ and $9 \cdot 2$; $7 \cdot 10$ and $9 \cdot 10$
3. Add \$15 and \$11; $15d$ and $11d$; $15 \cdot 2$ and $11 \cdot 2$.
4. Add 3 in., 4 in., and 8 in.; $3i$, $4i$, and $8i$.
5. 7 mi. + 14 mi.
6. $7m + 14m$.
7. $7 \cdot 2 + 14 \cdot 2$.
8. 6 bu. + 4 bu. + 9 bu.
9. $6b + 4b + 9b$.
10. $6 \cdot 7 + 4 \cdot 7 + 9 \cdot 7$.
11. How do you proceed to add similar monomials?
12. \$1 + \$17.
13. $d + 17d$.
14. $x^2 + 17x^2$.
15. $ab + 15ab$.
16. $x^2y^2 + 27x^2y^2$.
17. $abc + 32abc$.
18. Add -7, -3, and 11; $-7x$, $-3x$, and $11x$.
19. $-x + 2x$.
20. $-y + 13y$.
21. $-x - 2x - x$.
22. $-7x - 4x - x$.
23. $-2ab + 3ab + 4ab - 7ab + 10ab$.
24. $-21xyz + 30xyz + 2xyz - 3xyz$.
25. $2\sqrt{2} + 3\sqrt{2} + 4\sqrt{2} + 5\sqrt{2} + 10\sqrt{2} + 3\sqrt{2}$.
26. $4p + 5p$.
27. $4(a + b) + 5(a + b)$.
28. $5(a - b) + 2(a - b)$.
29. $4(a + b) - 2(a + b)$.
30. $-3(a - b) + 7(a - b)$.
31. $-2\sqrt{a + b} + 3\sqrt{a + b}$

46. Algebraic Sum. The result obtained by adding two or more numbers considered with respect to their signs as well as their values is called their *algebraic sum.*

Thus the algebraic sum of 2 and -3 is -1, although $2+3=5$.

47. Addition of Monomials. *To add similar monomials, find the algebraic sum of the coefficients of the common factor and prefix this sum to the common factor.*

For this purpose we may consider an expression like $27(a-b)$ as a monomial, as in Ex. 18, below. In case a letter has no coefficient, 1 is understood. Thus $5x+x=6x$.

To add dissimilar monomials, write the terms one after the other, each with its proper sign.

Thus the sum of a, $2b$, and $-c$ is $a+2b-c$.

Exercise 29. Addition of Monomials

Examples 1 to 11, oral — Examples 12 to 19, written

1. Add 2 ft. and 3 ft.; $2f$ and $3f$; $2\cdot 25$ and $3\cdot 25$.

2. Add 3 yd., 2 ft., and 7 in., expressing the result as a compound number. Add $3y$, $2f$, and $7i$; $3a$, $2b$, and $7c$.

3. Add a, $-2b$, and $3c$; x, $-2y$, and $3z$; m, $-2n$, and $3p$.

4. Add $-2a$, b, and $4c$; $-2x$, y, and $4z$; $-3x$, y, and z.

5. Add $5a$, $6b$, and $-7c$; $5p$, $6q$, and $-7r$; $7a$, $4b$, and c.

Add the following:

6. a, $-a$.
7. $-a$, a.
8. $19a$, $-3x$, $3x$.
9. $-4a$, $42x$, $4a$.
10. $17a$, $-19x$, $19x$.
11. $-17a$, $-19x$, $17a$.
12. $18a^2b$, $36a^2b$, $-7a^2b$.
13. $2\sqrt{a}$, $13\sqrt{a}$, $15\sqrt{a}$.
14. $16\sqrt{m}$, $-23\sqrt{m}$, $\sqrt{m}$.
15. $92abc$, $37abc$, $-75abc$.
16. $15mn^2$, $-7mn^2$, $16mn^2$.
17. $36pqr$, $49pqr$, $-17pqr$.
18. $27(a-b)$, $38(a-b)$, $33(a-b)$, $-17(a-b)$.
19. $24(x^2+y^2)$, $17(x^2+y^2)$, $-15(x^2+y^2)$, $46(x^2+y^2)$.

Exercise 30. Addition of Polynomials

Examples 1 to 9, oral — Examples 10 to 17, written

1. Add 2 ft. 3 in. and 3 in.; $2f + 3i$ and $3i$.

2. Add 7 yd. 2 ft. and 6 yd.; $7y + 2f$ and $6y$.

Add the following:

3.

$$\begin{array}{l} 6 \text{ ft.} + 7 \text{ in.} \\ \underline{2 \text{ ft.} + 3 \text{ in.}} \end{array}$$

4.

$$\begin{array}{l} 6f + 7i \\ \underline{2f + 3i} \end{array}$$

5.

$$\begin{array}{l} 6 \cdot 5 + \ 7 \cdot 8 \\ \underline{2 \cdot 5 + \ 3 \cdot 8} \\ (?) \cdot 5 + (?) \cdot 8 \end{array}$$

6.

$$\begin{array}{l} 2 \text{ rd.} + 8 \text{ ft.} + 9 \text{ in.} \\ \underline{4 \text{ rd.} + 3 \text{ ft.} - 2 \text{ in.}} \end{array}$$

7.

$$\begin{array}{l} 2r + 8f + 9i \\ \underline{4r + 3f - 2i} \end{array}$$

8.

$$\begin{array}{l} 2 \cdot 3 + 8 \cdot 7 + 9 \cdot 4 \\ \underline{4 \cdot 3 + 3 \cdot 7 - 2 \cdot 4} \end{array}$$

9.

$$\begin{array}{l} x^2 + 10xy + 15z^2 \\ \underline{x^2 + \ 4xy - 10z^2} \end{array}$$

10.

$$\begin{array}{l} 3w + 2x + 3y + 4z \\ \underline{2w - 3x + 2y + 3z} \end{array}$$

11.

$$\begin{array}{l} x^2 + 10y + 7z + 9 \\ \underline{x^2 - \ 3y + 4z - 9} \end{array}$$

12.

$$\begin{array}{r} 7^2 + \ 5 \cdot 7 \cdot 3 + \ 3^2 \\ \underline{7^2 - \ 2 \cdot 7 \cdot 3 + \ 3^2} \\ (?) \cdot 7^2 + (?) \cdot 7 \cdot 3 + (?) \cdot 3^2 \end{array}$$

13.

$$\begin{array}{l} x^4 + x^2y^2 + y^4 + x^2 + y^2 \\ \underline{x^4 - x^2y^2 + y^4 - x^2 - y^2} \end{array}$$

14.

$$\begin{array}{l} a + 3b - c + d - e + f \\ \underline{a - 3b + c - d + e - f} \end{array}$$

15.

$$\begin{array}{l} a^3 - 3a^2b + 3ab^2 - b^3 \\ \underline{a^3 + 3a^2b + 3ab^2 + b^3} \end{array}$$

16.

$$\begin{array}{l} x^3 + 3x^2y + 3xy^2 + y^3 \\ \underline{-x^3 + 3x^2y - 3xy^2 + y^3} \end{array}$$

17. If $f(x) = x^3 + 2x^2 - 7x + 1$ and $F(x) = 2x^3 + x^2 + 10x + 2$, what does $f(x) + F(x)$ equal?

B1

48. Addition of Polynomials. It is evident from the preceding exercise that

To add polynomials we write similar terms in the same column and add these terms, writing their sums as a polynomial.

49. Check. An operation that tends to prove the correctness of another operation is called a *check* upon that operation.

Thus in addition we check by adding in the other direction.

50. Checks in Algebra. One of the best checks on the operations in algebra is the substitution of any values we please for the letters, as in the following example, where we let $x = 1$, $y = 1$, and $z = 1$.

Operation	Check
$2x + 3y - 4z + 6$	$2 + 3 - 4 + 6 = 7$
$3x - 7y + 8z - 3$	$3 - 7 + 8 - 3 = 1$
$5x - 4y + 4z + 3$	$5 - 4 + 4 + 3 = 8$

Here we have simply put 1 in place of x, y, and z, in the addends and in the sum, and we have $7 + 1 = 8$. Therefore the work *checks*.

We may have an error in spite of this check, as would be the case if we should write $5y - 4x + 4z + 3$ instead of $5x - 4y + 4z + 3$, or if we should make an error in computation in the check. In case of doubt, especially in case of exponents, use other values than 1.

Exercise 31. Addition of Polynomials

Examples 1 to 3, oral — Examples 4 to 28, written

1. Add $2x + y$ and $3x + y$; $2x + y$ and $3x - y$.
2. Add $7x + 3y$ and $3x + 7y$; $7x + 3y$ and $-3x - 7y$.
3. Add $a + b + c$ and $a - b + c$; $a - b - c$ and $a + b + c$.

Add the following and check the results:

4. $x^2 + xy + y^2$, $x^2 - 2xy + y^2$, $x^2 + 2xy + y^2$.
5. $x^2 + y^2$, $x^2 - y^2$, $2x^2 + y^2$, $x^2 + 2y^2$, x^2, $3y^2$.
6. $x^3 + x^2y + xy^2 + y^3$, $2x^3 - 3x^2y + 7xy^2 - 6y^3$.

Add the following and check the results:

9. $5x^2+3x+7,\ 6x^2-3x+7,\ x+2.$
8. $7x^2-9x+2,\ 6x^2-8x+3,\ x-9.$
9. $2x^2+3x-9,\ 4x^2-6x-8,\ x+4.$
10. $3x^2+5x+6,\ 7x^2-7x+9,\ x-4.$
11. $4x^2-9x-8,\ 2x^2-x+1,\ x^2+3x-1.$
12. $-x^2+2x+7,\ -x^2-x+9,\ 3x^2+2x+3.$
13. $-5x^2-7x+9,\ -10x^2-14x+13,\ 2x^2-8.$
14. $a^2+ab+b^2,\ a^2-ab+b^2,\ -a^2+ab-b^2,\ a^2+b^2.$
15. $a^3+a^2+a+1,\ a^3-a^2+a-1,\ a^3+a^2-a+1,\ -3a^3.$
16. $p^2+q^2+4,\ p^2-q^2+7,\ -p^2+q^2-8,\ -p^2-q^2-3.$
17. $m^2n+mn^2+7,\ m^2n+mn^2-7,\ m^2n-2mn^2,\ -3m^2n.$

Simplify the following by combining like terms:

18. $a^2+3b^2-4c^2+2d^2-7+3a^2-4b^2+c^2-2d^2+7.$
19. $a^3+3a^2b+3ab^2+b^3-a^3+3a^2b-3ab^2-b^3.$
20. $x^3+2x^2+3x-1+4x+3x^2+1-x^3-5x^2-7x.$
21. $m^2n+mn^2+mn-3mn^2-mn+4m^2n.$
22. Add $3t^3+5t^2+4t+2$ and $2t^3+3t^2+2t+5$; also 3542 and 2325. What do the polynomials equal if $t=10$?

23. How much is 2 ft. + 3 ft.? a ft. + b ft.? $2f+3f$? $af+bf$? $ax+bx$? $am+bm$? $2\cdot5$ and $3\cdot5$? $a\cdot5$ and $b\cdot5$?

We may think of a ft. and b ft. as $(a+b)$ ft. Similarly, $af+bf=(a+b)f$. That is, we do not know the numerical value of the coefficients a and b, so we *indicate* their sum.

Add the following, indicating the sums of the coefficients:

24. p yd. + q yd.; $py+qy$; $pm+qm$; $p\cdot5+q\cdot5$.
25. $am+bm$; $ax+bx$; $axy+bxy$; $a\cdot2\cdot3+b\cdot2\cdot3$.
26. $m\sqrt{a}+n\sqrt{a}$; ma^2+na^2; $m\sqrt{a+b}+n\sqrt{a+b}$.
27. $ax+bx+cx$; $am^2+bm^2+cm^2$; $a\sqrt{m}+b\sqrt{m}+c\sqrt{m}$.
28. $x\sqrt{2}+y\sqrt{2}-z\sqrt{2}$; $x\sqrt{a-b}+y\sqrt{a-b}-z\sqrt{a-b}$.

51. Equations involving Addition. We have already studied equations involving addition, subtraction, multiplication, and division. We shall use all of these processes as necessary, but shall now consider some special features in addition.

Required to solve the equation $x + 3x + 4x = 64$.

Since $x + 3x + 4x = 64$,

therefore $8x = 64$, by uniting terms,

and $x = 8$, by dividing equals by 8.

Check. Substituting 8 in the *original equation*, we have

$$8 + 3 \cdot 8 + 4 \cdot 8 = 8 + 24 + 32 = 64.$$

Exercise 32. Equations involving Addition

Examples 1 to 8, oral — Examples 9 to 13, written

1. Solve the equation $x + 2x = 9$. How will you check?
2. Solve the equation $x + 4x = 25$. Check the result.

Solve the following and check the results:

3. $x + 7x = 72$.
4. $2x + 5x = 49$.
5. $3x + 2x = 75$.
6. $x + 2x + 3x = 6$.
7. $2x + x + 4x = 14$.
8. $3x + 3x + 3x = 27$.

9. A rectangle is twice as long as wide. If w represents the width, represent the length in terms of w. Represent the perimeter in terms of w. Draw the figure.

10. A rectangle is twice as long as wide. The perimeter is 12 in. What is the width? the length?

11. A rectangle is three times as long as wide. The perimeter is 24 in. What is the width? the length?

12. A rectangular field is four times as long as wide. The perimeter is 100 rd. What is the width? the length?

13. A rectangular box is twice as wide as deep, and twice as long as wide. If the sum of the twelve edges is 28 in., what is the depth? the width? the length?

CHAPTER V

SUBTRACTION

Exercise 33. Subtraction of Similar Monomials

Examples 1 to 17, oral — Examples 18 to 26, written

1. How much is 7 ft. $-$ 4 ft.? $7f-4f$? $7\cdot 2-4\cdot 2$?
2. How much is 10 mi. $-$ 3 mi.? $10m-3m$? $10a-3a$?
3. How much is 16 yd. $-$ 9 yd.? $16y-9y$? $16\cdot 3-9\cdot 3$?
4. How much is 27 rd. $-$ 14 rd.? $27r-14r$? $27\cdot 5-14\cdot 5$?

State the value of:

5. \$30 $-$ \$12.
6. $30d-12d$.
7. $30a-12a$.
8. $15x-7x$.
9. $15ax-7ax$.
10. $15\sqrt{a}-7\sqrt{a}$.
11. $29a-11a$.
12. $29x^2-11x^2$.
13. $29\sqrt{xy}-11\sqrt{xy}$.

14. How do you subtract one monomial from a similar monomial?

15. How much is a ft. $-$ b ft.? a mi. $-$ b mi.? $ax-bx$?

16. From 7° subtract 6°; from 7° subtract 7°; from 7° subtract 8°. What must be added to 6 to make 7? to 7 to make 7? to 8 to make 7?

17. How many degrees from $-1°$ to $+7°$? How much is $7°-(-1°)$? $7-(-1)$? $7a-(-a)$?

State the value of:

18. $4.7a-3.9a$.
19. $5.1x^2-2.8x^2$.
20. $7.2mn-3.9mn$.
21. $3x-2\frac{1}{3}x$.
22. $13\frac{3}{4}x-5\frac{1}{2}x$.
23. $12\frac{1}{2}y-7\frac{1}{8}y$.
24. $7°-(-3°)$.
25. $7d-(-3d)$.
26. $7x-(-3x)$.

52. Subtraction of Monomials. It is evident from the preceding exercise that

To subtract a monomial from a similar monomial, we find the difference between the coefficients of the common factor and multiply this difference by the common factor.

Thus $9a - 4a = 5a$, $ax - bx = (a - b)x$, and $mx^2 - 4nx^2 = (m - 4n)x^2$.

In case a letter has no coefficient expressed, 1 is understood as usual. Thus $5x - x = 4x$.

If the monomials are dissimilar we indicate the subtraction.

Thus, as we may write 7 ft. − 3 in., so we write $7f - 3i$, or $7x - 3y$, or $7\sqrt{a+b} - 3\sqrt{x+y}$.

Exercise 34. Subtraction of Monomials

Examples 1 to 3, oral — Examples 4 to 16, written

1. What must be added to 3 to make 7? to $3a$ to make $7a$? to $3a^2b$ to make $7a^2b$? How much is $7a^2b - 3a^2b$?

2. What must be added to $-3°$ to make $7°$? to $-3x$ to make $7x$? to $-3\sqrt{a}$ to make $7\sqrt{a}$? How much is $7a - (-3a)$?

3. What must be added to $-10°$ to make $4°$? to $-10x$ to make $4x$? How much is $4x - (-10x)$?

State the value of:

4. $30\sqrt{a} - 17\sqrt{a}$.
5. $5.1x^2 - 3.9x^2$.
6. $7\frac{1}{2}m - 3\frac{2}{3}m$.
7. $7\frac{1}{3}m - 5\frac{1}{2}m$.
8. $8\frac{2}{3}q - 7\frac{3}{4}q$.
9. $9a - 9\frac{3}{4}a$.
10. $7.1ab - 3.8ab$.
11. $9.3a^2x - 5.7a^2x$.
12. $a^2b^2 - ya^2b^2$.

13. From the sum of $4a$ and $7a$ subtract $-5a$.

14. From the sum of $5xy$ and $9xy$ subtract $-12xy$.

15. From the sum of $7ab$ and $8ab$ subtract the sum of $-2ab$, $15ab$, and $-9ab$.

16. From the sum of $6x^2$, $9x^2$, and $-3x^2$, subtract the sum of $17x^2$, $-6x^2$, $-9x^2$, and $3x^2$.

53. Subtraction as the Inverse of Addition. We may think of subtraction as the operation by which we find the number that added to the subtrahend will equal the minuend.

Thus, because 3 added to 4 makes 7, therefore $7 - 4 = 3$.
Because 7 added to -3 makes 4, therefore $4 - (-3) = 7$.
Because 5° added to $-9°$ makes $-4°$, therefore $-4° - (-9°) = 5°$.

We therefore have the following cases in subtraction:

$4 + 5 = 9$, therefore $9 - 5 = 4$;
$4 + (-5) = -1$, therefore $-1 - (-5) = 4$;
$-4 + 5 = 1$, therefore $1 - 5 = -4$;
$-4 + (-5) = -9$, therefore $-9 - (-5) = -4$.

Therefore, *to subtract one quantity from another, find a quantity that added to the subtrahend will equal the minuend.*

If the student will think of this when he subtracts, he will rarely be troubled in the matter of signs. If he is confused, he should, as already stated, remember that in taking a smaller number from a larger one the result must be positive, and in taking a larger number from a smaller one the result must be negative.

Exercise 35. Subtracting Monomials

Examples 1 to 16, oral — Examples 17 to 26, written

1. $3 - 4$.	9. $5ax - 3ax$.	17. $73a - 19a$.
2. $3 - (-4)$.	10. $5ax - (-3ax)$.	18. $37a - (-19a)$.
3. $-3 - 4$.	11. $7m^2 - 4m^2$.	19. $-71a^2 - 39a^2$.
4. $-3 - (-4)$.	12. $7m^2 - (-4m^2)$.	20. $-71a^2 - (-39a^2)$
5. $3a - 4a$.	13. $-6c - 7c$.	21. $51ab - 27ab$.
6. $3a - (-4a)$.	14. $-6c - (-7c)$.	22. $62ab - (-35ab)$.
7. $-3a - 4a$.	15. $-9x - 10x$.	23. $ax^2 - (-bx^2)$.
8. $-3a - (-4a)$.	16. $-9x - (-10x)$.	24. $a^2x^2y^2 - (-b^2x^2y^2)$.

25. From the sum of $3x$ and $7x$ subtract $5x - 2x$.

26. From the difference between $37.5ax$ and $29.3ax$ subtract the difference between $42.9ax$ and $40.7ax$.

Exercise 36. Subtracting Polynomials

Examples 1 to 11, oral — Examples 12 to 25, written

1. From 4 ft. 8 in. subtract 2 ft. 3 in.

2. From 8 ft. − 4 in. subtract 2 in.; subtract 6 ft. 2 in.

3. If the thermometer registers + 2° and the temperature falls 4°, what does it then register?

4. If the thermometer registers − 2° and the temperature falls 4°, what does it then register?

Subtract:

5.

7 lb. 4 oz.
3 lb. 2 oz.

6.

$7x + 4y$
$3x + 2y$

7.

$7 \cdot 3 + 4 \cdot 2$
$3 \cdot 3 + 2 \cdot 2$

8.

$9m + 48f$
$4m + 20f$

9.

$9a^2b + 48c$
$4a^2b + 20c$

10.

15 ft. − 4 in.
8 ft. + 2 in.

11.

$15x - 4y$
$8x + 2y$

12.

$8a + 4b$
$3a + 2b$

13.

$8a + 4b$
$5a + 4b$

14.

$8 \cdot 9 + 4 \cdot 5$
$5 \cdot 9 + 4 \cdot 5$

15.

$15m + 7n$
$9m + 6n$

16.

$15p + 7q$
$12p + 7q$

17.

$15p + 7q$
$14p + 8q$

18.

$20x - 8y$
$13x + 9y$

19.

$17x^2y^2 + 15xy$
$12x^2y^2 + 14xy$

20.

$23a^2b - 15b^2c$
$17a^2b + 16b^2c$

21.

$23 \cdot 10 - 15 \cdot 3$
$17 \cdot 10 + 16 \cdot 3$

22.

$71ax + mby$
$47ax + nby$

23.

$29x^2y + axy^2$
$30x^2y + bxy^2$

24.

$xm^2n - 21mn^2$
$ym^2n + 15mn^2$

25.

$ax^2y^2 + cxy$
$bx^2y^2 + dxy$

54. Subtracting Polynomials. From the preceding exercise we see that

To subtract one polynomial from another, we arrange similar terms under one another and subtract these terms separately.

For example, subtract $4a^2 - 3ab + 1$ from $6a^2 - 8 - 9ab$. Rearranging, we have the following:

Operation	Check
$6a^2 - 9ab - 8$	$6 - 9 - 8 = -11$
$4a^2 - 3ab + 1$	$4 - 3 + 1 = \quad 2$
$2a^2 - 6ab - 9$	$2 - 6 - 9 = -13$

Here the work is checked by letting $a = 1$ and $b = 1$. If a check upon the exponents is desired, use other values than 1 for a and b.

Exercise 37. Subtracting Polynomials

Examples 1 to 6, oral — Examples 7 to 13, written

1. From $4a + 3b$ take $2a + 3b$.
2. Subtract $6m - 3n$ from $7m - 3n$.
3. From $9x - 7y + 1$ take $10x - 8y + 1$.

Subtract and check:

4.	7.	10.
$12ab - 15cd$	$x^2 + xy + y^2$	$a^3 + 3a^2 + 4a$
$11ab - 15cd$	$x^2 - xy + y^2$	$a^3 - 3a^2 - 4a$
$1ab - cd$	$2xy$	$2a^3$
5.	**8.**	**11.**
$16ax - 17ay$	$a^2 - 2ab + b^2$	$4a^2 + 3a - 4$
$14ax - 18ay$	$a^2 + 2ab - b^2$	$2a^2 - 4a + 7$
$2ax + ay$	$2a^2$	$2a^2 + 7a - 11$
6.	**9.**	**12.**
$19mn - 15pq$	$2x^2 - 4xy + 7y^2$	$6x^2 + 5x - 7$
$14mn - 25pq$	$2x^2 + 4xy - 7y^2$	$4x^2 - 7x - 9$
$5mn + 10pq$	$-8xy + 14y^2$	$2x^2 - 2x - 2$

13. From $4a^3 + 7a^2b - 6ab^2 - 8b^3$ take $3a^3 - 9ab^2 + 7b^3 - 6a^2b$

Exercise 38. Subtracting Polynomials

Examples 1 to 8, oral — Examples 9 to 32, written

1. From a take $-a$; from $a+b$ take $a-b$.
2. From $-b$ take b; from $a-b$ take $a+b$.

Subtract:

3. a^2-b^2 from a^2+b^2.
4. a^2+b^2 from a^2-b^2.
5. $-a^2-b^2$ from a^2+b^2.
6. $ax-1$ from $ax+1$.
7. $ax+1$ from $ax-1$.
8. $1-ax$ from $ax-1$.
9. From $7x-2y+3$ take $4x+2y-3$.
10. From $9a+3b-4c$ take $8a+4b-7c$.
11. From $6a^2-9b^2+7c^2$ take $5a^2-8b^2-6c^2$.
12. From $17ab-19cd+25ef$ take $20ab-30cd+40ef$.

Rearrange the terms properly and subtract:

13. $a^2+c^2-b^2$ from $a^2+b^2-c^2$.
14. $a^2-3c^2+7b^2$ from $6b^2-7c^2-9a^2$.
15. $ax+cz-by$ from $ax+2by-3cz$.
16. $a^2b^2+m^2n^2-x^2y^2$ from $3m^2n^2+4x^2y^2-5a^2b^2$.
17. $a^2+b^2-c^2+4d^2$ from $3a^2-7d^2-9c^2$.
18. From $3a$ ft. $+2b$ in. take a ft. $+b$ in.
19. From $3af$ take af; from $3af+2bi$ take $af+bi$.
20. From $xf+yi$ take $af+bi$; from $xf+y$ take $wf+zy$.

If $A=2a^2+3ab-b^2$, $C=-a^2+5ab$,
$B=a^2-3ab+b^2$, $D=-5a^2-7ab-9b^2$,

find the expression for:

21. $A-B$.
22. $C-D$.
23. $A-C$.
24. $A-D$.
25. $A+B-C$.
26. $A+C-B$.
27. $B+C-A$.
28. $B-A-C$.
29. $A+B+C+D$.
30. $A+B-C+D$.
31. $A-B+C-D$.
32. $A-B-C-D$

55. Problems in Percentage. Problems in percentage are often more easily solved by algebra than by arithmetic. For example:

1. If 15% of a number is 9165, what is the number?

Solution by Arithmetic

Since 15% of the number = 9165,
therefore 1% of the number = $\frac{1}{15}$ of 9165, or 611,
and the number = 100 × 611, or 61,100.

Solution by Algebra

Let x represent the number.
Then $0.15x = 9165$,
and $x = 61{,}100$, by dividing both members by 0.15.

2. What number increased by $66\frac{2}{3}\%$ of itself equals 275?

Solution by Arithmetic

Since 100% of the number = the number,
therefore $66\frac{2}{3}\%$ of the number = the increase,
and $166\frac{2}{3}\%$, or $\frac{5}{3}$, of the number = 275.
Therefore $\frac{1}{3}$ of the number = $\frac{1}{5}$ of 275, or 55,
and the number = 3 × 55, or 165.

Solution by Algebra

Let x represent the number.
Then $x + 0.66\frac{2}{3}x = 275$,
and $x = 165$, by dividing both members by $1.66\frac{2}{3}$.

3. After deducting 10% from the marked price of some goods, a dealer sold them for \$13.50. What was the marked price?

Let x represent the *number* of dollars of marked price.
Then $x - 0.10x = 13.50$,
or $0.90x = 13.50$.
Therefore by dividing equals by 0.90, $x = 13.50 \div 0.90$
$= 15$.
Therefore the marked price was \$15.

Check. \$15 − 10% of \$15 = \$15 − \$1.50 = \$13.50.

Exercise 39. Percentage

Examples 1 to 10, oral — Examples 11 to 24, written

1. A number less 10% of itself is what per cent of itself?
2. A number plus 10% of itself is what per cent of itself?

State the per cent of x in the following:

3. $x + 0.05x$. 5. $x + 20\%x$. 7. $x + 0.50x$. 9. $x + 75\%x$.
4. $x - 0.05x$. 6. $x - 20\%x$. 8. $x - 0.50x$. 10. $x - 75\%x$.

11. What number less 10% of itself equals 72?
12. What number less 17% of itself equals 166?
13. \$435 is 6% of what sum of money?
14. \$19.75 is 5% of what sum of money?
15. \$38.25 is $4\frac{1}{2}\%$ of what sum of money?
16. What is the number of which 14.4 is $66\frac{2}{3}\%$?
17. What is the sum of which \$41.25 is 15%?
18. What is the sum of which $33\frac{1}{3}\%$ is \$2.25?
19. A certain number increased by $12\frac{1}{2}\%$ of itself equals 819. What is the number?
20. A boy now weighs 84 lb., which is 12% more than he weighed a year ago. How much did he weigh then?
21. A certain school gained 15% this year over the number last year. It now has 161 pupils. How many had it last year?
22. A dealer saved \$1968 this year from his store. This is 18% less than he saved last year. How much did he save last year?
23. A dealer was obliged to sell some damaged furniture at 10% less than cost. He sold it for \$85.50. How much did it cost? How much did he lose?
24. A farmer sold his milk to a factory where he received credit for 1045 lb. of butter fat. If his milk tested 3.8% butter fat, how many pounds of milk did he sell?

56. Removal of Parentheses preceded by the Positive Sign. If to \$4 we add \$3 + \$2, we have in all \$4 + \$3 + \$2, or \$9. It is evidently of no consequence whether we add \$2 to the sum of \$4 and \$3, or add \$5 to \$4.

Hence $\$4 + (\$3 + \$2) = \$4 + \$3 + \$2$.

Similarly, $5¢ + (2¢ + 7¢) = 5¢ + 2¢ + 7¢$,

$$a + (b + c) = a + b + c,$$

and $$a + (b - c) = a + b - c.$$

If an expression inclosed within parentheses is preceded by the sign +, the parentheses may be removed without any change in the signs of the terms.

Similarly, we may have any other sign of aggregation.

Thus $7 + \overline{3 + 4} = 7 + 3 + 4$, and $12 + [5 - 2] = 12 + 5 - 2 = 15$.

Exercise 40. Removal of Parentheses

Examples 1 to 10, oral — Examples 11 to 18, written

1. How much is \$4 plus the sum of \$3 and \$9? How much is the sum of \$4, \$3, and \$9?

Remove the parentheses and simplify the results:

2. $7 + (9 + 6)$.
3. $7 + (9 - 6)$.
4. $8 + (15 + 5)$.
5. $8 + (15 - 5)$.
6. $4a + (3a + a)$.
7. $5a + (7a + 3a)$.
8. $7a + (7a - 3a)$.
9. $6x^2y^2 + (9x^2y^2 - x^2y^2)$.

10. How much is the sum of 7 and 3 increased by the sum of 4 and 2? How much is the sum of 7, 3, 4, and 2?

Remove the parentheses and simplify the results:

11. $(2a + 3a) + (7a + a)$.
12. $(2a + 3a) + (7a - a)$.
13. $(2a - 3a) + (7a + a)$.
14. $(2a - 3a) + (7a - a)$.
15. $(6x + 9y) + (5x - 3y)$
16. $a - b + (a + b)$.
17. $a^2 - b^2 + (a^2 + b^2)$.
18. $5a + 17 + (5a - 17)$.

57. Removal of Parentheses preceded by the Negative Sign. If we subtract $b + c$ from a, and $b - c$ from a, we have the following results:

$$\begin{array}{r} a \\ b+c \\ \hline a-b-c \end{array} \qquad \begin{array}{r} a \\ b-c \\ \hline a-b+c \end{array}$$

We therefore see that

$$a-(b+c)=a-b-c,$$

and

$$a-(b-c)=a-b+c.$$

If an expression inclosed within parentheses is preceded by the negative sign, the parentheses may be removed, provided the sign before each term is changed.

That is, $x+y-(x-y)=x+y-x+y=2y;$

and

$$\frac{3}{2}-\frac{5-4}{2}=\frac{3-5+4}{2}=\frac{2}{2}=1,$$

the fraction bar having the force of parentheses.

If $A=x^2+2x-1,$

and $B=x^2-7x-8,$

then $A-B=x^2+2x-1-x^2+7x+8=9x+7,$

and $B-A=x^2-7x-8-x^2-2x+1=-9x-7.$

Exercise 41. Removal of Parentheses

Examples 1 to 8, oral — Examples 9 to 48, written

1. Subtract $7-5$ from 10. Subtract 7 from 10 and add 5.
2. Subtract $7a-5a$ from $10a$. How much is $10a-7a+5a$?

Remove the parentheses and simplify the results:

3. $20-(8+2)$.
4. $20-(8-2)$.
5. $30-(9-4)$.
6. $4a-(3a-a)$.
7. $7x-(4x-2x)$.
8. $9x-(5x-3x)$.

9. If $A = x^2 - 3x - 15$, and $B = x^2 - 9x - 11$, what is the value of $A - B$? of $B - A$?

10. In Ex. 9, what is the value of $x^2 - A$? of $x^2 + 7 - A$?

11. In Ex. 9, what is the value of $A - (x^2 - 5x + 7) - B$?

12. In Ex. 9, what is the value of $B - (x^2 + 9x - 6) - A$?

13. How much is $(17a + 15b) - (12a - b)$?

Remove the parentheses and simplify the results:

14. $12x - (9x + 7x)$.
15. $17y - (12y - 4y)$.
16. $39a - (15a + b)$.
17. $47a - (15b + 7a)$.
18. $56n + (27n - 4n)$.
19. $39ab + (ab + 1)$.
20. $56a^2 - (3a^2 + 7)$.
21. $48p + (27p - 2)$.
22. $63q - (49 + 4q)$.
23. $79x - (79 - 79x)$.
24. How much is $25a^2 - \overline{2a^2 + 1}$? $25a^2 + \overline{2a^2 + 1}$?

Remove the bars and simplify the results:

25. $4a - \overline{3a - a}$.
26. $7a - \overline{4a + 2a}$.
27. $8a + \overline{6a - 3a}$.
28. $9x - \overline{7x - 2x}$.
29. $ab + \overline{3 - ab}$.
30. $xy - \overline{7 + xy}$.
31. $pq - \overline{5 - pq}$.
32. $x^2y^2 - \overline{x^2y^2 - 1}$.

Remove the brackets and simplify the results:

33. $27a^2 - [4 - 27a^2]$.
34. $35ab - [5 + 35ab]$.
35. $42xy - [-xy + 42]$.
36. $75x^2y^2 + [-x^2y^2 + 7]$.
37. $80mn + [75 - 80mn]$.
38. $[35x + 1] - 1$.
39. $[42 - x] + x$.
40. $-[27 + ab] - ab$.
41. $-[xy - 24] + 24$.
42. $-[mn + 75] + 75$.

Remove the parentheses and brackets and simplify the results:

43. $[a + b] - (a - b)$.
44. $[a - b] - (a + b)$.
45. $[a + b] + (-a - b)$.
46. $-[a + b] - (-a - b)$.
47. $-[a - b] - (-a + b)$.
48. $[x^2 + y^2 - z^2] - (x^2 - y^2 + z^2)$.

58. Removal of Several Symbols of Aggregation. The symbols of aggregation most frequently used in algebra are the following:

Parentheses, as in $a-(b+c)$; *Brackets*, as in $p-[q-r]$;
Bar, or *vinculum*, as in $x-\overline{y-z}$; *Braces*, as in $m-\{m+n\}$.

Required to simplify the expression $10-(4-\overline{3-2})$.

$$\begin{aligned} 10-(4-\overline{3-2}) &= 10-(4-3+2) \\ &= 10-4+3-2 \\ &= 7. \end{aligned}$$

59. Insertion of Parentheses. It is also evident that

Two or more terms may be inclosed in parentheses preceded by a plus sign without changing the signs of the terms.

Two or more terms may be inclosed in parentheses preceded by a minus sign, provided the sign of each of the terms is changed.

Exercise 42. Removal of Symbols of Aggregation

Remove the symbols of aggregation and simplify:

1. $6x-(x-y)-y$.
2. $12ab-(6ab+3)-(5ab+3)$.
3. $10xy-(5xy+c)+(5xy-c)$.
4. $(a+b)-(b+c)-(c+d)-(d+e)-(e+a)$.
5. $3a-(b+c)+b-(c+d)+3c-(d-a)$.
6. $3x^2+2xy-7y^2-(3x^2-2xy+7y^2)$.
7. $3a^2-(4ab-b^2)-[a^2-(4ab+b^2)]$.
8. $5x^2-[x^2-(x^2-xy+y^2)+xy-3y^2]$.
9. $4a^3-(3a^2b-2ab^2)+3b^3-[a^3-(4a^2b-2ab^2+3b^3)]$.
10. $3a+6-(5a+7)-[a-3-(4a-\overline{2a-4})]$.
11. $3x^3-(2x^2y-2xy^2)+[y^3-4x^3+(2x^2y-3xy^2+3y^3)]$.
12. $2a^2-[(2a^2+2a)-(a^2-1)]+a^2-(a^2-a)$.
13. $5x^2y^2-[5-(x^2y^2+5xy)-x^2y^2]+(5-5x^2y^2)$.
14. In the expression $ax+bx+cx-px^2-qx^2-rx^2$ inclose the last three terms in parentheses without changing the value.

CHAPTER VI

MULTIPLICATION

Exercise 43. Multiplication of Monomials

Examples 1 to 20, oral — Examples 21 to 38, written

1. Multiply by 5: 2 mi.; 2 ft.; $2m$; $2f$; $2 \cdot 3$; $2x$.

2. Multiply by 7: 3 yd.; $3y$; 3 times a given number; $3n$

3. Multiply by 9: 4 in.; $4i$; \$4; $4d$; 4¢; $4c$; $4x^2y^2$; $4 \cdot 7$.

4. If the temperature is 3° below zero, what will it be when it is twice as much below zero?

5. How much is $2 \times (-3°)$? $2 \times (-4°)$? $2 \times (-3d)$? $2 \times (-4d)$? $2 \times (-4$ lb.)? $2 \times (-4l)$?

6. $2 \times 12a$.	10. $3 \times 20x$.	14. $4 \times 50m$.
7. $2 \times 12ab$.	11. $3 \times 20xy$.	15. $4 \times 50m^3$.
8. $2 \times (-12)$.	12. $3 \times (-20)$.	16. $4 \times (-50)$.
9. $2 \times (-12a^2)$.	13. $3 \times (-20x^2)$.	17. $4 \times (-50\sqrt{a})$.

18. How do you multiply a monomial by a positive integer?

19. How may $a \times a$ be written more briefly? $a \times a \times a$? $a \times a^2$? $a^2 \times a$? $a \times a^3$? $a^2 \times a^2$? $a^3 \times a^2$?

20. State a rule for multiplying a^3 by a^4.

21. $3 \times 4a^2$.	27. $7 \times 17m^2$.	33. $15 \times 17a$.
22. $a \times 4a^2$.	28. $7m \times 17m^2$.	34. $15a \times 17a$.
23. $3a \times 4a^2$.	29. $8 \times 36x^3$.	35. $15a^3 \times 17a^5$.
24. $5 \times 6x^3$.	30. $8x \times 36x^3$.	36. $27a^3 \times 17a^6$.
25. $x^2 \times 6x^3$.	31. $9x^2 \times 36x^2$.	37. $35a^4 \times 27a^5$.
26. $5x^2 \times 6x^3$.	32. $9x^2 \times 48x^3$.	38. $56y^4 \times 92y^4$.

60. Laws of Exponents. Since a^2 means $a \cdot a$, and a^3 means $a \cdot a \cdot a$, we see that

$$a^2 \cdot a^3 = a \cdot a \times a \cdot a \cdot a = a^{2+3} = a^5.$$

Furthermore, $$a^m \cdot a^n = a^{m+n}.$$

For $a^m = a$ taken m times as a factor,

and $a^n = a$ taken n times as a factor.

Therefore $a^m \cdot a^n = a$ taken $m + n$ times as a factor.

In multiplying monomials, the exponent of any letter in the product is equal to the sum of the exponents of that letter in the factors.

Since $(a^2)^3$ means $a^2a^2a^2$, or a^6, and $(a^m)^n$ means $a^m a^m a^m \cdot \cdot \cdot$ (n times), therefore

$$(a^m)^n = a^{mn}.$$

That is, $(a^3)^5 = a^{15}$, $(x^7)^3 = x^{21}$, and so on.

61. Law of Signs. The law of signs in multiplication is given on page 34. Briefly stated it is as follows:

In multiplication, two like signs produce plus, two unlike signs produce minus.

That is, $+ a \cdot (+ b) = + ab$, $+ a \cdot (- b) = - ab$,

$- a \cdot (- b) = + ab$, $- a \cdot (+ b) = - ab$.

Therefore $2x^2y \cdot 5x^3y^2 = 10x^5y^3$,

$2xy \cdot (- 5x^2y^2) = - 10x^3y^3$.

Evidently, therefore, *the product of an even number of negative factors is positive, and the product of an odd number of negative factors is negative.*

62. Degree of a Term. The number of literal factors in a term is called the *degree of the term.*

Thus a^2 is of the second degree, a^3 is of the third degree, a^m is of the mth degree, and a^{m+3} is of the $m + 3$ degree. Similarly, x^2y^3 is of the fifth degree, because $xxyyy$ has five literal factors.

We may, however, speak of x^2y^3 as of the second degree in x, or as of the third degree in y, if we mention specifically the letter.

63. Degree of a Polynomial. The degree of the term that is of the highest degree in a polynomial is called the *degree of the polynomial.*

Thus $a^2x^2 + bx + c$ is of the fourth degree, because a^2x^2 is of the fourth degree. It is, however, of the second degree in x, because a^2x^2 is of the second degree in x.

Similarly, $a^4x^2 + 1$ is of the sixth degree, but it is of the second degree in x and of the fourth degree in a.

64. Homogeneous Polynomial. A polynomial of which all the terms are of the same degree is said to be *homogeneous.*

Thus $x^3 + 3x^2y + 3xy^2 + y^3$ is homogeneous, because every term is of the third degree.

Similarly, $x^4 + x^2y^2 + y^4$ is a homogeneous trinomial of the fourth degree.

65. Multiplication of Monomials. As we have seen in the exercise on page 59, to find the product of two monomials we may proceed as follows:

Find the product of the numerical coefficients, writing after this product the letters, each letter having an exponent equal to the sum of its exponents in the factors.

It will be found better to write the product in this order: the sign of the product; the product of the numerical coefficients; the letters in their alphabetical order, each with its proper exponent.

For example,

$$4ab \cdot 5ab = 4 \cdot 5 \cdot a \cdot a \cdot b \cdot b = 20a^2b^2;$$

$$3a^mb \cdot 4a^nb = 3 \cdot 4 \cdot a^m \cdot a^n \cdot b \cdot b = 12a^{m+n}b^2;$$

$$(-7x^7y^6)^2 = -7 \cdot (-7) \cdot x^7 \cdot x^7 \cdot y^6 \cdot y^6 = 49x^{14}y^{12};$$

$$-6abc \cdot (-7a^2b^2c^2) = 6 \cdot 7 \cdot a \cdot a^2 \cdot b \cdot b^2 \cdot c \cdot c^2 = 42a^3b^3c^3;$$

$$-5p^qr^s \cdot (-4p^2r^3) = 5 \cdot 4 \cdot p^q \cdot p^2 \cdot r^s \cdot r^3 = 20p^{q+2}r^{s+3}.$$

In practice the result should be written down rapidly, or stated orally, without the intermediate step here given.

In multiplying a by b, instead of saying "*a times b*" it is common to say "*a into b.*" The expression is a very old one, and the word "into" used in this sense has lost its original meaning, but the student will often hear it used in algebra.

Exercise 44. Multiplication of Monomials

Examples 1 to 45, oral — Examples 46 to 62, written

1. a^2a^9.
2. b^8b^8.
3. c^9c^4.
4. d^7d^7.
5. e^9e^8.
6. m^8m^{11}.
7. p^4p^{12}.
8. $a^{11}a$.
9. $a^{12}a$.
10. a^ma.
11. a^ma^4.
12. a^ma^p.
13. x^ax^b.
14. a^xa^y.
15. $a^{2x}a^x$.
16. $a^{3x}a^x$.
17. $a^{3x}a^{2x}$.
18. $ab \cdot ab$.
19. $a^2b \cdot a^2b$.
20. $a^2b^3 \cdot a^2b^3$.
21. $a^4b^7 \cdot a^5b^2$.
22. $p^9q^7 \cdot p^7q^9$.
23. $5\,a^7 \cdot 6\,a^5$.
24. $7\,a^8 \cdot 8\,a^7$.
25. $9\,x^m \cdot 7\,x$.
26. $9\,x^m \cdot 8\,x^2$.
27. $8\,x^m \cdot 5\,x^n$.
28. $-\,a^2 \cdot 3\,a^4$.
29. $-\,a^8 \cdot 5\,a^7$.
30. $-\,2\,a^4 \cdot 6\,a^6$.
31. $3\,a \cdot (-\,4\,a)$.
32. $3\,a \cdot (-\,4\,a^2)$.
33. $3\,a^m \cdot (-\,5\,a^n)$.
34. $ab \cdot (-\,ab)$.
35. $-\,ab \cdot ab$.
36. $-\,ab \cdot (-\,ab)$.
37. $x^2y^2 \cdot (-\,2\,xy)$.
38. $-\,3\,xy^3 \cdot (-\,7\,xy)$.
39. $-\,4\,x^2y^3 \cdot (-\,5\,x^2y)$
40. $-\,6\,x^2y^5 \cdot 10\,x^3y^3$.
41. $abc \cdot abc$.
42. $a^2b^2c^2 \cdot abc$.
43. $a^mbc \cdot a^nbc$.
44. $(a^2b^2c^2d^2)^3$.
45. $(p^3q^3r^5s^5)^2$.

46. If the circumference of a circle is πd, what is the sum of the circumferences of 7 circles of the diameter d?

47. If the circumference of a circle, πd, is multiplied by d, what is the result? What is the result if it is multiplied by $\frac{1}{2}d$?

48. If an edge of a cube is $4x$, what is the volume of the cube?

49. $-\,27\,x^m \cdot (-\,23\,x^7)$.
50. $-\,42\,x^{17} \cdot (-\,34\,x^9)$.
51. $-\,68\,p^m \cdot (-\,27\,p^n)$.
52. $-\,47\,p^x \cdot (-\,31\,p^y)$.
53. $-\,96\,p^x \cdot (-\,41\,p^{2x})$.
54. $(17\,x^2)^3$; $(96\,x^5)^3$.
55. $(24\,x^4)^2$; $(73\,x^7)^2$.
56. $37\,x^{4m} \cdot 67\,x^{9m}$.
57. $83\,x^{m+1} \cdot 75\,x^{m-1}$.
58. $43\,a^{p+4} \cdot 72\,a^{p-4}$.
59. $-\,49\,x^{a+b} \cdot 32\,x^{a-b}$.
60. $-\,26\,x^{a+3b} \cdot 34\,x^{a-5b}$.
61. $72\,a^{2x+1} \cdot 33\,a^{4x-2}$.
62. $34\,x^{a-7} \cdot (-\,42\,x^{4a+9})$.

66. Multiplication of a Polynomial by a Monomial. If we multiply 5 ft. 2 in. by 3, we have 15 ft. 6 in. In the same way, if we multiply 5 times one number and 2 times another number by 3, we have 15 times the first plus 6 times the second. That is,

5 ft. 2 in.	$5f + 2i$	$5x + 2y$
3	3	3
15 ft. 6 in.	$15f + 6i$	$15x + 6y$

In the same way we have the following:

Operation	Check
$a^2 - 2ab + 3b^2$	$1 - 2 + 3 = 2$
ab	$1 = 1$
$a^3b - 2a^2b^2 + 3ab^3$	$1 - 2 + 3 = 2$

It should be noticed that in algebra it is more convenient to write the multiplier at the left and work from left to right.

In this check we let $a = 1$ and $b = 1$. This checks the coefficients, where the error is most liable to occur, but it does not check the exponents, since any power of 1 is 1. If a check upon the exponents is desired, let $a = 2$ and $b = 2$, or take other values. In case either factor becomes zero in the check, use some other values for the letters.

It is also a good check to notice that if both multiplicand and multiplier are homogeneous, the product will also be homogeneous.

To multiply a polynomial by a monomial, multiply each term of the polynomial by the monomial and add the partial products.

Exercise 45. Multiplying by a Monomial

Examples 1 to 6, oral — Examples 7 to 12, written

1. $a(b + c)$.
2. $-a(b - c)$.
3. $3a^2(a + b)$.
4. $-3a^2(a - b)$.
5. $a(b + c - d)$.
6. $-a(b - c + d)$.
7. $2x^2(x^2 - 2xy + y^2)$.
8. $4x^3(x^2 - 3x + 27)$.
9. $-7ab(a^2 - 2ab + b^2)$.
10. $-9a^2b(a^2 - 2ab + b^2)$.
11. $35a^3x^5(a^7 - 3a^4x^2 + 2ax^5)$.
12. $-8x^ny^n(x^ny^n - 7xy - 13)$.

67. Multiplication of a Polynomial by a Polynomial. If we multiply 43 by 21, we multiply first by 1 unit and then by 2 tens, and then add the partial products, thus:

Multiplicand,	43	$40+3$
Multiplier,	21	$20+1$
Multiplying by 1 unit,	43	$40+3$
Multiplying by 2 tens,	86	$800+60$
Sum of partial products,	903	$800+100+3$

In a similar manner we multiply $3a^2-b$ by a^2+b, thus:

	Operation	Check
	$3a^2-b$	$3-1=2$
	a^2+b	$1+1=2$
Multiplying by a^2,	$3a^4-a^2b$	4
Multiplying by b,	$3a^2b-b^2$	
	$3a^4+2a^2b-b^2$	$3+2-1=4$

To multiply a polynomial by a polynomial, multiply the multiplicand by each term of the multiplier and add the partial products.

Exercise 46. Multiplying by a Monomial or Binomial

Examples 1 to 10, oral — Examples 11 to 20, written

1. $a(a+b)$.
2. $b(a+b)$.
3. $-4a(a-b)$.
4. $-7a^2(a-b)$.
5. $-9a^2(a^2-b^2)$.
6. $15(2xy+1)$.
7. $15(2xy-2)$.
8. $-8(4x^2y-3)$.
9. $-7(7x^2y^2-9)$.
10. $-12x(5x^m-3x^n)$.
11. $(a+b)(a-b)$.
12. $(a+b)(a+b)$.
13. $(a^2-b^2)(a-b)$.
14. $(a^2-b^2)(a^2+b^2)$.
15. $(a-7)(a^2-7a+1)$.
16. $(2x-1)(4x^2+x-1)$.
17. $(3x-2)(3x^2-4x+7)$.
18. $(a^2b^2+1)(a^2b^2+4)$.
19. $(abc-1)(a^2b^2c^2+5)$.
20. $(a^2b^2c-2)(a^2b^2c+2)$.

68. Arranging a Polynomial. A polynomial in which the exponents of a certain letter in the successive terms decrease from left to right is said to be *arranged according to the descending powers* of that letter.

Thus $x^3 + 3x^2 - 4x + 1$ is arranged according to the descending powers of x, and $an^2 + a^2n - a^3$ is arranged according to the descending powers of n.

Similarly, a polynomial may be arranged according to the *ascending* powers of a letter. Thus $a^3 - 3a^2b + 3ab^2 - b^3$ is arranged according to the ascending powers of b.

In multiplying, it simplifies the work if the polynomials are arranged according to the ascending or descending powers of some letter.

Multiply $4a^3 - 2a^2 + 7 - a$ by $3 + a^2 - 3a$.

OPERATION	CHECK
Rearranging, we multiply thus:	
$4a^3 - 2a^2 - a + 7$	$= 8$
$a^2 - 3a + 3$	$= 1$
$4a^5 - 2a^4 - a^3 + 7a^2$	8
$-12a^4 + 6a^3 + 3a^2 - 21a$	
$12a^3 - 6a^2 - 3a + 21$	
$4a^5 - 14a^4 + 17a^3 + 4a^2 - 24a + 21$	$= 8$

In the operation we multiply first by a^2, then by $-3a$, and finally by 3. We then add the partial products.

In the check we have let $a = 1$. We have then only to add the coefficients, having $4 - 2 - 1 + 7 = 8$, and $1 - 3 + 3 = 1$, and similarly for the product.

Multiply $x^2 - 3xy + 3y^2$ by $y^2 + x^2$.

Rearranging, we multiply thus:

$x^2 - 3xy + 3y^2$	$= 1$
$x^2 + y^2$	$= 2$
$x^4 - 3x^3y + 3x^2y^2$	2
$x^2y^2 - 3xy^3 + 3y^4$	
$x^4 - 3x^3y + 4x^2y^2 - 3xy^3 + 3y^4$	$= 2$

Exercise 47. Multiplying by a Polynomial

Examples 1 to 4, oral — Examples 5 to 50, written

1. Arrange $a^2 + a^3 + 1 + a$ for multiplying.

2. Arrange $x + 1 + x^4 - x^2 + 3x^5 + x^6 + x^3$ according to ascending powers of x.

3. Arrange $x^2 + y^2 + 2xy$ according to descending powers of x. How is it then arranged with respect to y?

4. Apply the check of $x = 1$, $y = 1$, and find if $x^2 + 3xy + y^2$ can be the product of $x + y$ and $x + y$.

Multiply and check:

5. $(p + q)(p + q)$.

6. $(x + y)(x + y)$.

7. $(m + n)(m + n)$.

8. $(a + b)(a + 2b)$.

9. $(a + 2b)(a - 3b)$.

10. $(x^2 + 1)(x^2 + 1)$.

11. $(p + q)(p - q)$.

12. $(x + y)(x - y)$.

13. $(x^2 + 1)(x^2 - 1)$.

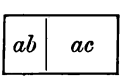
14. $(x^2 + y^2)(x^2 + 2y^2)$.

15. $(xy + 3)(xy + 4)$.

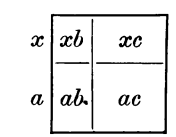
16. $(xyz + 1)(xyz - 5)$.

17. Since a rectangle with length b and width a has an area ab, show that these squares and rectangles illustrate the products indicated beneath:

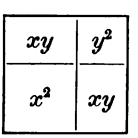
$a \times b = ab$



$a(b + c)$
$= ab + ac$



$(x + a)(b + c)$
$= xb + ab + xc + ac$



$(x + y)(x + y)$
$= (?)$

18. Multiply the product of $a + b$ and $a - b$ by $a^2 + b^2$.

19. Multiply the product of $x + y$ and $x + y$ by $x^2 - y^2$.

20. Multiply the product of $m + 7$ and $m - 7$ by $m^2 + 49$.

21. Multiply the product of $2 + y$ and $2 - y$ by $4 + y^2$.

Multiply and check:

22. $a^2 + 2ab + b^2$ by $a + b$.
23. $a^2 - 2ab + b^2$ by $a - b$.
24. $x^2 + 2xy + y^2$ by $x^2 + 2xy + y^2$.
25. $x^3 + 3x^2y + 3xy^2 + y^3$ by $x + y$.
26. $x^2 + 2xy + y^2$ by $x^2 - 2xy + y^2$.
27. $x^3 - x^2y + xy^2 - y^3$ by $x + y$.
28. $x^3 + x^2y + xy^2 + y^3$ by $x - y$.
29. $2x^3 + 3x^2y - 4xy^2 + 3y^3$ by $x^2 - 2xy + 4y^2$.
30. $4m^3 + 3m^2n - 7mn^2 + n^3$ by $m^2 + n^2$.
31. $6m^4 + 3m^2 + 2m + 1$ by $m^2 + 3m - 2$.
32. $7p^3 + 8p^2 + 6p - 4$ by $p^2 - 4p + 1$.
33. $8x^3 - 4x^2 + 7x - 6$ by $x^2 - 5x - 3$.
34. $a^2b^2 + 4ab - 7$ by $a^2b^2 - 7ab + 9$.
35. $a^2x^3 - ax^2 + x - 7$ by $a^2x^3 - x + 4$.
36. $a^2 + x^2 - ax + 5$ by $a^2 - x^2 + ax - 5$.
37. $a^3 + b^3 - a^2b + ab^2$ by $a^2 - ab + b^2$.
38. $5a^5 + 4a^4 + 3a^3 + 2a + 1$ by $2a + 1$.
39. $3x^2y - x^3 + y^3 - 4xy^2$ by $x^2 + y^2 + 4xy$.
40. $5m^3 + 7m^2 - 9m + 1$ by $m^2 - 4 + m$.
41. $x^3 - 3x^2 + 7 - 4x$ by $x^3 + 3 - 5x + 2x^2$.
42. $a^2b^2c^2 + a^3b^3c^3 - abc + 7$ by $a^2b^2c^2 + 1$.
43. $a^4x^4 + 1 - a^3x^3 - a^2x^2 - ax$ by $a^3x^3 - ax + 3$.
44. $25a^2 + 50ab + 1$ by $25a^2 - 50ab + 1$.
45. $a^2b^2x^2 - 4abx + 1$ by $a^2b^2x^2 + 4abx + 1$.
46. $a^n + a^{n-1} + a^{n-2}$ by a; also by a^2; also by $a + 1$.
47. $a^m + b^m$ by $a^m - b^m$, and this product by $a^{2m} + b^{2m}$.
48. $x^m - x^{m-1} + x^{m-2} - x^{m-3}$ by $x + 1$.
49. $x^{m+1} + x^m + x^{m-1} + x^{m-2}$ by $x + 1$.
50. $a^{2m} - 2a^mb^m + b^{2m}$ by $a^{2m} + 2a^mb^m + b^{2m}$.

Exercise 48. Review

Examples 1 to 6, oral — Examples 7 to 19, written

1. Add $2x + 1$ and $2x - 1$; also $ax + 1$ and $ax - 1$.
2. Add $3x + 4$ and $5x + 6$; also $ax + b$ and $cx + d$.
3. Subtract $a^2 - b^2$ from $a^2 + b^2$; also $a^2 - b^2$ from a^2.
4. Subtract $a^2 + 2ab + b^2$ from $a^2 - 2ab + b^2$.
5. Multiply $a + b$ by a; by b; by ab; by a^2.
6. Multiply $2a - b$ by a; by $-a$; by b; by $-b$.

Simplify:

7. $(a + b)(a - 3b) - (2a + b)(3a - b)$.
8. $(a + b + c)(a + b - c) - (a^2 + b^2 - c^2)$.
9. $(a + b + c)(a + b - c)(a - b + c)$.
10. $(a - b)(b - c) + (b - c)(c - d) + (c - d)(d - a)$.
11. $(x + 1)(x + 2) + (x + 2)(x + 3) - (2x + 3)(x - 7)$.
12. $(x + y)(y + z) - (z + w)(x + w) - (x + z)(y - w)$.
13. $m^2(x - y) - m^2(x + y) + m^2(y + z) + m^2(y - z)$.
14. $3[x - (y + z)] - 3[y - (x + z)] - 6(x - y + z)$.
15. $5[a + b - (a - b)] - 5b(1 - 2a) - 10ab$.
16. $2a^2 + 3ab - [a^2 + 2ab - (2a^2 + ab)] - (a^2 - ab)$.

17. If a certain number increased by 2 is multiplied by the number increased by 3, the product equals 16 more than the square of the number. What is the number?

18. If the number of students in a certain algebra class is increased by 4, and this sum is multiplied by the number increased by 5, the product equals 200 more than the square of the number. What is the number?

19. By multiplying $a + b$ by $a + b$ show that the square of the sum of two numbers equals the square of the first plus twice the product of the first and second, plus the square of the second. Apply this law to squaring $20 + 5$.

CHAPTER VII

DIVISION

Exercise 49. Division of Monomials

Examples 1 to 18, oral — Examples 19 to 34, written

1. Divide by 2: 4 mi.; $4m$; $4 \cdot 5$; $4x$; $4 \cdot 3 \cdot 5$; $4xy$.

2. Divide by 5: 10 lb.; $10l$; $10n$; $10n^2$; $10xy$; $10abc$.

3. Divide by 9: 27 rd.; $27r$; 36 in.; $36i$; 45 acres; $45a$.

4. If the temperature is 4° below zero, what will it be when it is half as much below zero? How much is $-4° \div 2$?

5. $64¢ \div 8$.	9. $-4° \div 2$.	13. $75ab \div 25$.
6. $64c \div 8$.	10. $-4x \div 2$.	14. $60a^2x \div 30$.
7. $64xy \div 8$.	11. $-4abc \div 2$.	15. $45p^2q^2 \div 15$.
8. $64\sqrt{a} \div 8$.	12. $-16x^2 \div 2$.	16. $90\sqrt{x} \div 45$.

17. How do you divide a monomial by a positive integer?

18. Divide \$8 by \$2; $8d$ by $2d$; 8 ft. by 2 ft.; $8f$ by $2f$; $8 \cdot 5$ by $2 \cdot 5$; 8 times any number by 2 times the number.

Divide:

19. $8x$ by $2x$.	27. $125a^2$ by $5a^2$.
20. $10x$ by $2x$.	28. $275x^2$ by $25x^2$.
21. $15y$ by $5y$.	29. $250p^2$ by $50p^2$.
22. $16z$ by $8z$.	30. $175m$ by $25m$.
23. $25m$ by $5m$.	31. $-350ab$ by $70ab$.
24. $36p$ by $4p$.	32. $-450pq$ by $25pq$.
25. $40a$ by $10a$.	33. $-750a^2x$ by $25a^2x$.
26. $75b$ by $25b$.	34. $-925abc$ by $25abc$.

69. Law of Exponents. Division being the inverse of multiplication, we have the following:

Since $a^2 \cdot a^3 = a^5$, therefore $a^5 \div a^3 = a^2$;

since $a^x \cdot a^y = a^{x+y}$, therefore $a^{x+y} \div a^y = a^x$,

or $a^m \div a^n = a^{m-n}$.

The exponent of any letter in the quotient is equal to the exponent of that letter in the dividend minus the exponent of that letter in the divisor.

That is, $a^m \div a^n = a^{m-n}$.

Likewise, $-21x^9y^4 \div 7x^5y = -3x^4y^3$.

In algebra, division is usually expressed in the fractional form, thus:

$$\frac{48a^5b^4c^3}{-4abc} = -12a^4b^3c^2.$$

70. Law of Signs. The law of signs is given on page 35. Briefly stated it is as follows:

In division, two like signs produce plus; two unlike signs produce minus.

That is, $\frac{4}{2} = 2$, $\frac{-4}{-2} = 2$, $\frac{-4}{2} = -2$, and $\frac{4}{-2} = -2$.

Exercise 50. Division of Monomials

Examples 1 to 5, oral — Examples 6 to 20, written

1. $\frac{a^2b^2}{ab}$.
2. $\frac{m^3n^4}{mn}$.
3. $\frac{x^7y^5}{xy}$.
4. $\frac{x^9y^8}{x^2y^2}$.
5. $\frac{-4a^5}{2a^3}$.
6. $\frac{11.1x^7}{3x^2}$.
7. $\frac{-9.6m^4n}{-0.4mn}$.
8. $\frac{135a^4b^3c}{5abc}$.
9. $\frac{96x^4y^4z^4}{-6xyz}$.
10. $\frac{1.21p^5q^5r}{1.1pq^2r}$.
11. $\frac{-119m^7n^5}{7m^5n^5}$.
12. $\frac{144m^4n^6}{-4m^4n}$.
13. $\frac{-175x^2y^4z^6}{-5x^2yz^5}$.
14. $\frac{-225m^2n^2p^4}{-25mn^2p}$.
15. $\frac{375a^{10}b^9c^3}{25ab^7c^3}$.
16. $\frac{-425p^8q^7r^6}{5p^6q^7r}$.
17. $\frac{603a^8b^7c^6}{-9ab}$.
18. $\frac{-504x^{10}y^9z}{3x^9y^8}$.
19. $\frac{25a^mb^mc^m}{-5abc}$.
20. $\frac{-36a^mb^mc^m}{-4a^nb^nc^n}$.

71. Division of a Polynomial by a Monomial. If we divide 10 ft. 8 in. by 2, we have 5 ft. 4 in. Similarly, we have

$2)\underline{10 \text{ ft. } 4 \text{ in.}}$ $4)\underline{40 \text{ yd. } 16 \text{ in.}}$ $9)\underline{9 \text{ tens} + 9}$

$5 \text{ ft. } 2 \text{ in.}$ $10 \text{ yd. } 4 \text{ in.}$ $1 \text{ ten} + 1$

$2)\underline{10f + 4i}$ $4)\underline{40 \cdot 3 + 16 \cdot 5}$ $9)\underline{9t + 9}$

$5f + 2i$ $10 \cdot 3 + 4 \cdot 5$ $t + 1$

That is, to divide a polynomial by a monomial,

Divide each term of the dividend by the divisor and add the partial quotients.

If we divide zero by any number, the result is zero. Division by zero has as yet no meaning; the expression $a \div 0$ will be discussed later.

Exercise 51. Division of a Polynomial by a Monomial

Examples 1 to 9, oral — Examples 10 to 17, written

1. Divide by 2: 8 ft. 6 in.; $8f + 6i$; 8 tens + 6; $8t + 6$; 86.
2. Divide by 3: 9 mi. 15 rd.; $9m + 15r$; $9xy + 15xy$.
3. Divide by a: $ax + ay$; $abc + axy$; $a\sqrt{x} + a\sqrt{y}$; $a + a^2$.

Divide:

4. $px^2 + py^2$ by p.
5. $axy + amn$ by a.
6. $mpq - mxy$ by m.
7. $-ax + ay$ by a.
8. $-ax + ay$ by $-a$.
9. $am^2 - an^2 + ax^2$ by a.

Perform the indicated divisions:

10. $\dfrac{35p^4m + 75p^2m^3}{5p^2m}$.
11. $\dfrac{64a^2b^3c^4 - 48a^4b^3c^2}{-4a^2b^2c^2}$.
12. $\dfrac{75a^5b^4c^3 - 65a^3b^4c^5}{-5a^3b^3c^3}$.
13. $\dfrac{-81m^4n^3 + 108m^3n^4}{-9m^3n^3}$.
14. $\dfrac{25m^3p + 35m^4q - 15m^5r}{-5m^3}$.
15. $\dfrac{ab^2c^3 - a^3b^2c + a^2b^2c^2}{ab^2c}$.
16. $\dfrac{-4ab^2x - 6a^2by + 8abxy}{-2ab}$.
17. $\dfrac{-9mnx^3 + 15m^2n^2x^2 - 3mn}{-3mn}$.

72. Division of a Polynomial by a Polynomial. Since division is the inverse of multiplication, the process of division is best understood by first studying a case in multiplication.

Multiplicand,	$a^2 + 2ab + b^2$
Multiplier,	$a + b$
1st partial product, by a,	$a^3 + 2a^2b + ab^2$
2d partial product, by b,	$a^2b + 2ab^2 + b^3$
Product by $a + b$,	$a^3 + 3a^2b + 3ab^2 + b^3$

Reversing this, required to divide $a^3 + 3a^2b + 3ab^2 + b^3$ by $a^2 + 2ab + b^2$. The work is more conveniently arranged by placing the divisor at the right, thus:

Dividend,	$a^3 + 3a^2b + 3ab^2 + b^3$	$a^2 + 2ab + b^2$	Divisor
Product by a,	$a^3 + 2a^2b + ab^2$	$a + b$	Quotient
Remainder,	$a^2b + 2ab^2 + b^3$		
Product by b,	$a^2b + 2ab^2 + b^3$		

Dividing a^3, the first term of the dividend, by a^2, the first term of the divisor, we have a, the first term of the quotient.

Multiplying the divisor by a, we have $a^3 + 2a^2b + ab^2$, which is subtracted from the dividend, leaving $a^2b + 2ab^2 + b^3$ still to be divided.

Proceeding as before, and dividing a^2b by a^2, the next term of the quotient is b.

Multiplying as before, we have $a^2b + 2ab^2 + b^3$, which is subtracted, leaving no remainder.

We therefore see that in division we proceed as follows:

Arrange both dividend and divisor in ascending or descending powers of some common letter.

Divide the first term of the dividend by the first term of the divisor and write the result for the first term of the quotient.

Multiply the entire divisor by the first term of the quotient and subtract the result from the dividend.

If there is a remainder, consider it as a new dividend and proceed as before.

73. Illustrative Problems. The following typical problems should be carefully studied:

1. Divide $x^2 - 5x - 84$ by $x + 7$.

OPERATION

$$\begin{array}{r|l} x^2 - 5x - 84 & x + 7 \\ \hline x^2 + 7x & x - 12 \\ \hline -12x - 84 & \\ -12x - 84 & \\ \hline \end{array}$$

CHECK

$$\frac{-88}{8} = -11$$

We may check the result (1) by carefully reviewing the work, (2) by multiplying the quotient by the divisor, the product being the dividend, or (3) by substituting some convenient values for the letters.

Applying the second of these checks, we can easily show that $(x+7)(x-12) = x^2 - 5x - 84$.

Applying the third check, letting $x = 1$, we have $1 - 5 - 84 = -88$, $1 + 7 = 8$, $-88 \div 8 = -11$; and the quotient, $x - 12$, becomes -11 also.

2. Divide $a^3 + b^3$ by $a - b$.

OPERATION

$$\begin{array}{r|l} a^3 + b^3 & a - b \\ \hline a^3 - a^2b & a^2 + ab + b^2 + \dfrac{2b^3}{a-b} \\ \hline a^2b + b^3 & \\ a^2b - ab^2 & \\ \hline ab^2 + b^3 & \\ ab^2 - b^3 & \\ \hline 2b^3 & \end{array}$$

CHECK

$$a = 2,\ b = 1$$

$$\frac{9}{1} = 7 + \frac{2}{1}$$

In the check we cannot let $a = b$ because this would make the divisor zero. We therefore let $a = 2$ and $b = 1$. Then the dividend is 9, the divisor 1, and the quotient $7 + \frac{2}{1}$, or 9.

Since we cannot divide by zero, therefore, in checking the work we use some value for the letters that shall not make the divisor zero.

It is also a good check to notice that if both dividend and divisor are homogeneous, the quotient is also homogeneous.

Exercise 52. Division of a Polynomial by a Polynomial

Examples 1 to 5, oral — Examples 6 to 30, written

1. In dividing $x^3 + 3x^2y + 2xy^2$ by $x^2 + 2xy$, what is the first term of the quotient?

2. How do you check the work in division?

State the first term of the quotient:

3. $x^2 - 5x + 6$ divided by $x - 3$; by $x - 2$.

4. $8m^2 - 10mn - 3n^2$ divided by $4m + n$; by $2m - 3n$.

5. $p^3 - 8p - 3$ divided by $-p + 3$; by $p^2 + 3p + 1$.

Divide the following, checking the results:

6. $a^2 + 3a + 2$ by $a + 1$.

7. $b^2 + 5b + 4$ by $b + 1$.

8. $c^2 + 8c + 12$ by $c + 2$.

9. $d^2 - 2d - 15$ by $d + 3$.

10. $e^2 + 11e + 28$ by $e + 4$.

11. $f^2 - 3f - 40$ by $f + 5$.

12. $g^2 - 11g + 18$ by $g - 2$.

13. $h^2 - 13h + 30$ by $h - 3$.

14. $a^2 - 5ab - 6b^2$ by $a + b$.

15. $a^2 - 6ab + 5b^2$ by $a - b$.

16. $c^2 + cd - 6d^2$ by $c + 3d$.

17. $p^2 - 9pq + 20q^2$ by $p - 5q$.

18. $x^2 + xy - 20y^2$ by $x - 4y$.

19. $p^2 + pq - 20q^2$ by $p + 5q$.

20. $x^2 + 11xy + 28y^2$ by $x + 7y$.

21. $p^2 + 11pq + 29q^2$ by $p + 4q$.

Arrange according to the descending powers of x, and divide:

22. $x^3 + x^4 - 16x - 4 - 9x^2$ by $4 + x^2 + 4x$.

23. $6 + x^4 - 12x + 11x^2 - 5x^3$ by $3 - 3x + x^2$.

24. $2x^4 + 9x - 12 - 5x^3 - 7x^2$ by $1 - x + x^2$.

25. $-7x^3 - 10x^2 + x^4 - 3 + 25x$ by $x + x^2 - 3$.

26. $3x^3 + x^4 - 6x - 5x^2 + 5$ by $3x + x^2 - 2$.

27. $6x - x^3 + x^4 - 9$ by $x^2 + 3 - x$; by $x^2 + x - 2$.

28. $x^3 + 8x^2 + 9x + 2$ by $x^2 + 7x + 2$.

29. $x^4 + 5x^3 + 7x^2 - 3x - 22$ by $x^2 + 2x - 3$.

30. Divide $2h + 8t + 8u$ by $h + 4t + 4u$; 288 by 144.

CHAPTER VIII

SIMPLE EQUATIONS WITH ONE UNKNOWN QUANTITY

74. Simple Equation. An equation involving the first power and no higher power of the unknown quantity is called a *simple equation.*

Such an equation is also said to be of the *first degree.* It is also called, particularly in higher mathematics, a *linear equation.*

75. Identity. An equation that is true for any value whatsoever of any letter is called an *identity.*

Thus $(a + b)^2 = a^2 + 2ab + b^2$. This is true if $a = 3$, $b = 4$, or if the letters have any other values whatsoever.

An equation involving only numbers, like $2 + 3 = 5$, is also called an identity.

An identity is sometimes expressed by the symbol $\equiv$, as in $a + b \equiv b + a$, read "$a + b$ is identical to $b + a$."

76. Satisfying an Equation. The quantity that, substituted for the unknown quantity, reduces an equation to an identity is said to *satisfy* the equation.

This quantity is the value of the unknown quantity. Thus if $x + 7 = 9$, then $x = 2$, and 2 satisfies the equation.

If $ax = a^2$, and we consider x as the unknown quantity, we see that $x = a$, and a satisfies the equation.

77. Solving an Equation. To find the value or values of the unknown quantity that will satisfy the equation is to *solve* the equation.

Thus if $x + 3 = 5$, then $x = 2$; for if we put 2 for x, the two members of the equation become $2 + 3$, and 5, which are the same in value. In other words, the equation then becomes an identity.

78. Root. A value of the unknown quantity that satisfies an equation is called a *root* of the equation.

Thus in $x - 4 = 9$, 13 is the root of the equation.

Simple equations have only one root. We shall later study equations that have more than one root. For example, the equation $x^2 - 3x + 2 = 0$ is satisfied if $x = 1$, and also if $x = 2$.

79. Transposition. We have already learned that we may subtract equals from equals and have equals left (§ 18). We have also seen that this amounts to taking a quantity from one side of the equation, changing the sign, and putting it on the other side.

Taking a quantity from one side of an equation, changing its sign, and putting it on the other side is called *transposition*

For example, if $2x - 4 = 8 + x$,

we may subtract x from both sides and have

$$x - 4 = 8.$$

We may then subtract -4 from both sides (or add $+4$ to both sides) and have $x = 12$.

The word "transpose" is therefore unnecessary, since we can use "subtract" or "add" in its stead, but it is a word that is commonly employed by many teachers of algebra.

80. Numerical Equation. An equation containing no letters except those representing a root is called a *numerical equation.* The following is the method of solving such an equation.

Solve the equation $4x - 7 + x = 10 - 3x - 1$.

Combining the terms, $5x - 7 = 9 - 3x$.

Subtracting -7, $5x = 16 - 3x$.

Subtracting $-3x$, $8x = 16$.

Dividing by 8, $x = 2$.

Check. $4 \times 2 - 7 + 2 = 10 - 3 \times 2 - 1$,

or $3 = 3$.

The solution may be shortened, thus:

Combining terms, $5x - 7 = 9 - 3x$.

Adding 7 and $3x$, $8x = 16$.

Dividing, $x = 2$.

Exercise 53. Numerical Equations

Examples 1 to 15, oral — Examples 16 to 32, written

1. Is $x + 3 = 4$ an identity? Give your reason.

2. Is $x + 3 = 3 + x$ an identity? Give your reason.

3. What value of x satisfies the equation $x - 4 = 5$? Prove it.

4. What is the root of the equation $x + 7 = 12$? Prove it.

5. What right have you to transpose -3 in the equation $x - 3 = 9$? Explain why the sign is changed.

Solve the equations:

6. $x - 4 = 6$.
7. $x + 4 = 6$.
8. $6 - x = 4$.
9. $x + 5 = 9$.
10. $x - 5 = 9$.
11. $9 - x = 5$.
12. $x + 7 = 15$.
13. $x + 15 = 7$.
14. $x - 15 = 7$.

15. How will you begin to solve the equation $4x + 5 - 2x = 15$? What will be your second step? your third step?

Solve the equations:

16. $5x - 7 + 3x = 9$.
17. $7x - 4 - 5x = 10$.
18. $9x + 7 - 37 = 3x$.
19. $9x - x - 44 = 4x$.
20. $7 - x + 7x = 31$.
21. $8 + 2x - 19 = x$.
22. $5 - x - 8 + 6x = 17$.
23. $3 - 7x - 14x - 1 = 23$.
24. $5x - 7 + 7x - 5 = 36$.
25. $9\frac{1}{2}x + 7 + 3\frac{1}{2}x = 20$.

26. If to 5 times a certain number we add 2, the sum is 20 diminished by 4 times the number. What is the number?

Solve the equations:

27. $5x - 7 + 3x + 6 + 8x = 13 + 6x + 40$.
28. $7x - 2 + 9x - 3 + 5x = 4 + x + 31$.
29. $8x + 7 - 2x + 5 = 4x + 12 - (x - 30)$.
30. $5x - (4 - 9x) - 6 = 2x - (7 - 3x) + 15$.
31. $3\frac{3}{8}x + 2\frac{1}{2} - (5\frac{1}{8}x - 7\frac{3}{4}) = 7\frac{1}{8}x - (5\frac{3}{4} - 9\frac{3}{8}x) + 26\frac{1}{4}$.
32. $2\frac{7}{8}x + 3\frac{3}{4} + (5\frac{1}{4}x - 9\frac{1}{2}) - 12\frac{5}{8}x = 2\frac{1}{4}x + 5\frac{3}{8} - (7\frac{3}{4}x - 12)$.

81. Problems relating to Numbers. The following problems should be studied carefully before attempting the next exercise.

1. Five times a certain number equals 75. Find the number.

Let x = the number.

Then $5x$ = five times the number.

But 75 = five times the number.

$\therefore 5x = 75.$

Dividing by 5, $x = 15.$

Therefore the required number is 15.

Check. Substituting in the statement of the problem, $5 \times 15 = 75.$

2. Five times a certain number equals the number increased by 24. Find the number.

Let x = the number.

Then $5x$ = five times the number,

and $x + 24$ = the number increased by 24.

$\therefore 5x = x + 24.$

Subtracting x, $4x = 24.$

Dividing by 4, $x = 6.$

Therefore the required number is 6.

Check. Substituting in the statement of the problem, $5 \times 6 = 6 + 24.$

3. The sum of two numbers is 39, and one of them is three more than five times the other. Find the numbers.

Let x = the smaller number.

Then $5x + 3$ = the larger number.

Since their sum is 39, therefore

$(5x + 3) + x = 39.$

Combining, $6x + 3 = 39.$

Subtracting 3, $6x = 36.$

Dividing by 6, $x = 6.$

Then $5x + 3 = 5 \times 6 + 3$

$= 33.$

Therefore the required numbers are 6 and 33.

Check. Substituting in the statement of the problem,

$6 + 33 = 39,$

and $33 = 5 \times 6 + 3.$

After a little practice solutions of this nature may be shortened.

Exercise 54. Problems in Numerical Equations

Examples 1 to 5, oral — Examples 6 to 14, written

1. If you represent a number by x, how will you represent four times the number? five more than four times the number?

2. If one book costs x dollars, how much will seven books cost? nine books? fifteen books? n books?

3. If a man bought 15 horses at d dollars each, how much did he pay for all the horses? for a third of them?

4. How will you represent four times a number, increased by seven? four times a number, decreased by six?

5. How will you represent the excess of a number over 45? the excess of 45 over a number? a number increased by 45?

6. Four increased by three times a certain number equals 19. Find the number.

7. Four times a certain number is diminished by three, the result being 29. Find the number.

8. If from five times a certain number seven is subtracted, the result is 33. Find the number.

9. Eight times a certain number equals 45 diminished by the number. Find the number.

10. Nine times a certain number equals 90 diminished by six times the number. Find the number.

11. Twelve times a certain number equals 160 diminished by eight times the number. Find the number.

12. Fifteen times a certain number equals 87 diminished by fourteen times the number. Find the number.

13. Nine times a certain number is diminished by 4.61, the result being 76.39. Find the number.

14. Ten times a certain number is diminished by six, the result being 60 more than four times the number. Find the number.

82. Problems relating to Per Cents. The following problems are typical of those met in percentage.

1. A man receives 6% on some money invested and adds \$60 to the amount received, thus making \$300. How much has he invested?

Let x = the number of dollars invested.

Then $0.06x$ = the number of dollars received,

and $0.06x + 60$ = the number of dollars stated, or 300.

$\therefore\ 0.06x + 60 = 300.$

Subtracting 60, $0.06x = 240.$

Dividing by 0.06, $x = 4000.$

Therefore he has \$4000 invested.

Check. 6% of \$4000 = \$240, and \$240 + \$60 = \$300.

It should be observed that we do not let x equal the *money*, but we let x = the *number* of dollars. Then when we find that $x = 4000$, we know that this is the *number* of dollars, and that \$4000 is the sum invested. We are thus relieved of the necessity of considering the dollar sign in the solution. In general, in algebra, this plan is pursued, all the numbers being considered abstract.

2. What per cent above cost must a man mark his goods so as to allow a discount of 20% and still make a profit of 20%?

The question is to find what per cent the marked price is of the cost, and then to find how much this is above cost.

Let c = the number of dollars of cost.

Then $1.20c$ = the number of dollars of selling price.

Let m = the number of dollars of marked price.

Then $0.80m$ = the number of dollars of selling price.

Therefore $0.80m = 1.20c.$

Dividing by 0.80, $m = 1.50c.$

That is, the goods must be marked 50% above cost.

Check. If from $1.50c$ we take 20% of it, we have left 80% of $1.50c$, or $1.20c$. This $1.20c$ is 20% more than c.

For example, if the goods cost \$80 the dealer marks them at $1.50 \times \$80$, or \$120. The selling price is then 20% less, or \$96, and this is 20% more than \$80.

Exercise 55. Problems relating to Per Cents

Examples 1 to 12, oral — Examples 13 to 20, written

1. Six is six per cent of what number?
2. Three is six per cent of what number?
3. Six is three per cent of what number?

Solve:

4. $6\%\ x = 18$.
5. $5\%\ x = 15$.
6. $4\%\ x = 12$.
7. $3\%\ x = 9$.
8. $2\%\ x = 6$.
9. $12\%\ x = 24$.
10. $x + 6\%\ x = 106$.
11. $x + 5\%\ x = 105$.
12. $x + 5\%\ x = 210$.

13. If 18% by weight of wheat is lost in grinding it into flour, how many pounds of wheat are used if the loss is 360 lb.?

14. From Ex. 13, how many pounds of wheat are needed to produce 820 lb. of flour?

15. From Ex. 13, how many pounds of wheat are needed to produce 779 lb. of flour?

16. A dealer sells a suit of clothes for \$32.20. The suit cost him \$28. What is his per cent of gain?

17. A suit of clothes was sold for \$30.80, after a discount of 12% was allowed on the marked price. What was the marked price?

18. The profits on a business this year are 22% more than they were last year. This year they are \$6344. What were they last year?

19. Water in freezing expands 10% of its volume. How many cubic inches of water will make 508.2 cu. in. of ice? Allowing 231 cu. in. to a gallon, how many gallons of water are needed?

20. At what per cent above cost must a dealer mark his goods so that he can deduct 25% and still make a profit of 25%? What will be the marked price if the goods cost the dealer \$800? What will be the selling price?

83. Problems relating to Mixtures. In various kinds of business it becomes necessary to mix certain ingredients, and the quantities are often found easily by algebra.

How much water must be added to 12 qt. of a 25% solution of ammonia to reduce it to a 10% solution?

Let x = the number of quarts added.

Then $12 + x$ = the total number of quarts.

Then $10\%(12 + x)$ = the number of quarts of ammonia.

But 25% of 12 = the number of quarts of ammonia.

$\therefore 10\%(12 + x) = 25\%$ of 12.

Combining, $1.2 + 10\%x = 3.$

Subtracting 1.2, $10\%x = 1.8.$

Dividing by 10%, $x = 18.$

Therefore 18 qt. of water must be added.

Check. 25% of 12 qt. = 3 qt., the amount of ammonia.

12 qt. + 18 qt. = 30 qt., the amount of the mixture.

Furthermore, 3 qt. is 10% of 30 qt., as required.

Exercise 56. Problems relating to Mixtures

All work written

1. How much water must be added to a gallon of a 22% solution of ammonia to reduce it to an 11% solution?

2. How much water must be added to 63 gal. of alcohol 95% pure to reduce it to a $66\frac{2}{3}\%$ solution?

3. How many pounds of water must be added to 320 lb. of a 4% solution of salt to reduce it to a $2\frac{1}{2}\%$ solution? How many gallons, allowing 8.35 lb. of water to a gallon?

4. How much water must be added to a pint of a 10% solution of a certain medicine to reduce it to a 2% solution?

5. How much water must be added to a barrel of vinegar 80% pure to reduce it to a 60% solution?

6. How much vinegar must be added to a barrel of vinegar 60% pure to make it an 80% solution?

84. Problems relating to Motion. Algebra is useful in solving problems relating to motion.

1. A train leaves Pittsburgh for the West at the same time that one leaves for the East. The former travels at the average rate of 42 mi. an hour and the latter at the rate of 38 mi. an hour. In how many hours will they be 240 mi. apart?

Let x = the number of hours.

In 1 hr. they are $(42 + 38)$ mi. apart, or 80 mi. apart.

In x hr. they are $x \cdot 80$ mi. apart, or $80x$ mi.

But they are 240 mi. apart.

$$\therefore 80x = 240.$$

Dividing by 80, $x = 3$.

Therefore in 3 hr. they will be 240 mi. apart.

Check. $3 \times (48 + 32)$ mi. $= 240$ mi.

2. A man starts from a certain place and travels on his bicycle at the rate of 16 mi. an hour. Forty-five minutes later an automobile starts after him at the rate of 24 mi. an hour. How long will it take the automobile to overtake him?

In 45 min. the bicycle has gone $\frac{3}{4}$ of 16 mi., or 12 mi.

The automobile goes 24 mi. an hour while the bicycle goes 16 mi., the automobile thus gaining on the bicycle 8 mi. an hour.

Let x = the number of hours required.

Since the automobile gains 8 mi. an hour,

$8x$ = the number of miles gained in x hr.

But 12 = the number of miles to be gained.

$$\therefore 8x = 12.$$

Dividing by 8, $x = 1\frac{1}{2}$.

Therefore the automobile will overtake the bicycle in $1\frac{1}{2}$ hr.

Check. In $1\frac{1}{2}$ hr. the automobile will travel $1\frac{1}{2} \times 24$ mi., or 36 mi., and the bicycle will travel $1\frac{1}{2} \times 16$ mi., or 24 mi. Adding 12 mi., the start of the bicycle, 24 mi. + 12 mi. = 36 mi.

In all problems relating to motion the average rate is to be understood unless the contrary is stated. In other words, the motion referred to in such problems is to be taken as uniform.

Exercise 57. Problems relating to Motion

Examples 1 to 4, oral — Examples 5 to 9, written

1. At 3 mi. an hour, how far will a man travel in 4 hr.? in $\frac{1}{2}$ hr.? in t hours? in $(a + b)$ hours?

2. At r miles an hour, how far will an aëroplane go in t hours? in $2t$ hours? in $\frac{1}{2}$ hr.?

3. If two men travel in opposite directions, one at the rate of 3 mi. an hour and the other at the rate of 40 mi. an hour, how far apart will they be in 1 hr.? in 2 hr.? in h hours?

4. If two men start from Chicago and New York at the same time and travel toward one another, the first at the rate of a miles an hour and the second at the rate of b miles an hour, how much nearer one another will they be in t hours?

5. Two men start from Denver at the same time, one traveling south 35 mi. an hour, and the other north 38 mi. an hour. How many miles apart will they be in 3 hr.? In how many hours will they be 292 mi. apart?

6. Two men start from the same place, one going east and the other going west, the former traveling twice as fast as the latter. In 4 hr. they are 300 mi. apart. Find the rate of each.

7. Two men start from the same place, one to the north and the other to the south, the former traveling 5 mi. an hour faster than the latter. In 2 hr. they are 150 mi. apart. Find the rate of each.

8. In Ex. 7 suppose the former travels 5 mi. an hour slower than the latter, and that in 2 hr. they are also 150 mi. apart. Find the rate of each.

9. A man leaves a friend at the railway station and starts to drive north just as his friend leaves on the train for the south. A half hour later they are 20 mi. apart. If the man drives one fourth as fast as the train travels, what is the rate of each?

85. Miscellaneous Problems. The following solutions should be studied before attempting the next exercise.

1. The sum of three consecutive numbers is 48. Find the numbers.

Let n = the middle number.
Then $n - 1$ = the smallest number,
and $n + 1$ = the largest number.
Adding, $3n$ = the sum.
Then $(n - 1) + n + (n + 1) = 48$,
whence $3n = 48$.
Dividing by 3, $n = 16$.
Therefore the numbers are 15, 16, 17.

Check. $15 + 16 + 17 = 48$.

It simplifies the solution a little to take $n - 1$, n, and $n + 1$ for the numbers, instead of n, $n + 1$, and $n + 2$, although the latter plan is also correct. In this case we should have

$$n + (n + 1) + (n + 2) = 48,$$
whence $3n = 45$,
and $n = 15$.
Therefore the numbers are 15, 16, 17.

2. A man is now three times as old as his son. Five years ago he was four times as old as his son. Find the age of each.

Let x = the number of years in the son's age.
Then $3x$ = the number of years in the father's age.
Also $x - 5$ = the number of years in the son's age 5 years ago,
and $3x - 5$ = the number of years in the father's age 5 years ago.
But five years ago the father was four times as old as the son.

$$\therefore 3x - 5 = 4(x - 5).$$

Multiplying, $3x - 5 = 4x - 20$.
Subtracting -5, $3x = 4x - 15$.
Subtracting $4x$, $-x = -15$.
Dividing by -1, $x = 15$, the number of years in the son's age,
and $3x = 45$, the number of years in the father's age.

Check. $45 = 3 \times 15$; five years ago they were 40 years and 10 years of age respectively, and $40 = 4 \times 10$.

Exercise 58. Miscellaneous Problems

Examples 1 to 6, oral — Examples 7 to 72, written

1. Solve the equations $5x + 7 = 42$; $5x - 7 = 43$.

2. Solve the equations $2x = 7$; $2x - 1 = 10$; $2x + 1 = 10$.

3. What number is one more than half itself?

4. A rectangle 5 in. wide has an area of 40 sq. in. What is its length?

5. A rectangular box 5 ft. long and 4 ft. wide has a capacity of 40 cu. ft. What is its depth?

6. A rectangular tank 5 ft. deep and 8 ft. wide has a capacity of 480 cu. ft. What is its length?

7. If it takes a steamer $5\frac{1}{2}$ da. to go 3355 mi., what is its average rate per day?

8. After playing 20 games a ball team had won three times as many as it had lost. How many games had it lost? How many had it won?

9. A freight train running 35 mi. an hour leaves a station three hours ahead of an express train that runs 50 mi. an hour. How soon will the express train overtake the freight?

10. The three angles of a triangle are together equal to 180°. In a certain right triangle one acute angle is three times as large as the other. Find the number of degrees in each angle.

11. The vertical angle of an isosceles triangle is 30°. Find the number of degrees in each of the two equal angles at the base.

12. A school building is in the form of a rectangle 60 ft. long and 49 ft. wide. An addition 35 ft. wide is to be made to cover half the area of the original building. How long is the addition?

13. If we can send a ten-word message to a certain place for 35¢, and a 24-word message for 63¢, what is the charge for each additional word above ten?

14. A miner secured 675 oz. of silver from some ore, this being $\frac{3}{4}\%$ of the weight of the ore. How much did the ore weigh?

15. The top of a tree 120 ft. high is broken off. The length of the part broken off is four times the length of the part left standing. Find the length of each part.

16. Three men buy a summer cottage for \$3000. The second pays twice as much as the first, and the third pays as much as the first two together. How much does each pay? If they used the cottage for three months, each occupying it alone, how much of the time should each be allowed to use it?

17. In 30 years from now a boy will be three times as old as he is at present. How old is he now?

18. Ten years ago a boy was one third as old as he is at present. How old is he now?

19. Four years ago a man was seven times as old as his son, and his son is now eight years old. Find the age of the father.

20. A man has \$9 in half dollars and quarters, having four times as many quarters as half dollars. How many coins of each kind has he?

21. A man has \$3.85 in quarters and dimes, having three times as many dimes as quarters. How many coins of each kind does he have?

22. The sum of the ages of a father and his son is 60 years, and the father's age is three years more than twice the age of his son. Find the age of each.

23. Some boys are laying out a circular running track that is to be just a quarter of a mile in length. What radius must they use in describing the circle? (Use $\pi = 3\frac{1}{7}$, as in § 15.)

24. Ten men agreed to build a camp together, but two decided not to take their share, each of the others then having to pay \$5 more for his share. Find the cost of the camp.

Given the following formulas, and using $3\frac{1}{7}$ for π wherever it occurs, find the value of the missing letter:

PARALLELOGRAM $a = bh$. § 5

	a	b	h
25.		$7\frac{1}{2}$	$9\frac{1}{2}$
26.		$4\frac{3}{4}$	$2\frac{1}{2}$
27.	75	15	
28.	161	23	
29.	319	29	
30.	341		11
31.	403		13
32.	437		19

CIRCUMFERENCE $c = 2\pi r$. § 15

	c	r
39.		7
40.		14
41.		$3\frac{1}{2}$
42.		21
43.	44	
44.	88	
45.	132	
46.	77	

SURFACE OF SPHERE $s = 4\pi r^2$

	s	r
53.		3
54.		5
55.		6
56.		10
57.	154	
58.	616	
59.	1386	
60.	2464	

CYLINDER $v = \pi r^2 h$

	v	r	h
33.		2	7
34.		7	2
35.		2	14
36.		7	15
37.	4400	10	
38.	1232	14	

AREA OF CIRCLE $a = \pi r^2$. § 16

	a	r
47.		2
48.		4
49.		7
50.	154	
51.	616	
52.	1386	

VOLUME OF SPHERE $v = \frac{4}{3}\pi r^3$

	v	r
61.		1
62.		2
63.		3
64.	$113\frac{1}{7}$	
65.	$1437\frac{1}{3}$	
66.	$4\frac{4}{21}$	

Given the area of a triangle (§ 6) $a = \frac{1}{2}bh$, *find* b *when:*

67. $a = 15$, $h = 5$. **68.** $a = 17.5$, $h = 5$.

From the same formula find h *when:*

69. $a = 8.25$, $b = 5.5$. **70.** $a = 10\frac{1}{16}$, $b = 3\frac{1}{8}$.

Given the area of a trapezoid (§ 13) $a = \frac{1}{2}(b + b')h$, *find:*

71. b' when $a = 40$, $b = 6$, and $h = 8$.

72. h when $a = 54$, $b = 8$, and $b' = 10$.

CHAPTER IX

SPECIAL PRODUCTS AND QUOTIENTS

86. Special Products and Quotients. There are certain products that are so often required in algebra that it is helpful to be able to write them at once without the labor of multiplying. In the same way there are certain quotients that we need to be able to write at once without taking the trouble to divide.

87. Square of the Sum or Difference of Two Numbers. If we multiply $a + b$ by $a + b$, the product is $a^2 + 2ab + b^2$, and if we multiply $a - b$ by $a - b$, the product is $a^2 - 2ab + b^2$, as is shown below.

$$\begin{array}{r} a + b \\ a + b \\ \hline a^2 + ab \\ ab + b^2 \\ \hline a^2 + 2ab + b^2 \end{array} \qquad \begin{array}{r} a - b \\ a - b \\ \hline a^2 - ab \\ - ab + b^2 \\ \hline a^2 - 2ab + b^2 \end{array}$$

The square of the sum of two numbers is the square of the first, plus twice their product, plus the square of the second.

That is, $(a + b)^2 = a^2 + 2ab + b^2$.

Therefore

$13^2 = (10 + 3)^2 = 10^2 + 2 \times 10 \times 3 + 3^2 = 169,$

$35^2 = (30 + 5)^2 = 900 + 2 \times 30 \times 5 + 25 = 1225.$

ab	b^2
a^2	ab
a	b

It is easily seen that the figure representing the square on $a + b$ is made up of a^2, ab, ab, and b^2, or $a^2 + 2ab + b^2$.

The square of the difference of two numbers is the square of the first, minus twice their product, plus the square of the second.

That is, $(a - b)^2 = a^2 - 2ab + b^2$.

Therefore $37^2 = (40 - 3)^2 = 40^2 - 2 \times 40 \times 3 + 3^2 = 1369.$

Exercise 59. Square of the Sum or Difference

Examples 1 to 46, oral — Examples 47 to 68, written

1. $(x+y)^2$.	**11.** $(x-y)^2$.	**21.** $(2a+b)^2$.	**31.** 11^2.
2. $(p+q)^2$.	**12.** $(p-q)^2$.	**22.** $(2a-b)^2$.	**32.** 12^2.
3. $(m+n)^2$.	**13.** $(m-n)^2$.	**23.** $(2a+1)^2$.	**33.** 21^2.
4. $(a+x)^2$.	**14.** $(a-x)^2$.	**24.** $(2a-1)^2$.	**34.** 22^2.
5. $(b+y)^2$.	**15.** $(b-2y)^2$.	**25.** $(4x+y)^2$.	**35.** 16^2.
6. $(a+1)^2$.	**16.** $(a-2)^2$.	**26.** $(5x-y)^2$.	**36.** 31^2.
7. $(a+2)^2$.	**17.** $(a-7)^2$.	**27.** $(x+6y)^2$.	**37.** 14^2.
8. $(x+3)^2$.	**18.** $(x-6)^2$.	**28.** $(x-7y)^2$.	**38.** 41^2.
9. $(4+n)^2$.	**19.** $(8-n)^2$.	**29.** $(2a+3)^2$.	**39.** 51^2.
10. $(5+y)^2$.	**20.** $(9-y)^2$.	**30.** $(2a-3)^2$.	**40.** 49^2.

41. Divide $x^2+2xy+y^2$ by $x+y$; x^2+2x+1 by $x+1$.

42. Divide $p^2-2pq+q^2$ by $p-q$; p^2-2p+1 by $p-1$.

43. Divide $m^2+2mn+n^2$ by $m+n$; $1+2n+n^2$ by $1+n$.

44. Divide $m^2-2mn+n^2$ by $m-n$; $1-2n+n^2$ by $1-n$.

45. How should you proceed if you wished to write the square of $5x+3y$ without multiplying in the usual way?

46. How should you proceed if you wished to write the square of $9x^3-7y^2$ without multiplying in the usual way?

47. $(2x+3)^2$.	**54.** $(12x^2y^2+1)^2$.	**61.** $(abc+7)^2$.
48. $(2x-3)^2$.	**55.** $(12x^2y^4-1)^2$.	**62.** $(abc-7)^2$.
49. $(7x+6)^2$.	**56.** $(15x^3y^3+2)^2$.	**63.** $(7-abc)^2$.
50. $(7x-6)^2$.	**57.** $(15x^3y^3-2)^2$.	**64.** $(a^3b^2c+12)^2$.
51. $(5x^2+9)^2$.	**58.** $(2-15x^3y^3)^2$.	**65.** $(a^3b^2c-12)^2$.
52. $(5x^2-9)^2$.	**59.** $(x^4y^3+x^3y^4)^2$.	**66.** $(12-a^3b^2c)^2$.
53. $(9-5x^2)^2$.	**60.** $(x^4y^3-x^3y^4)^2$.	**67.** $(7a^3b^2c+11)^2$.

68. Square $(a+b)+c$ as if $a+b$ were one term. Then remove the parentheses in the result.

88. Product of the Sum and Difference of Two Numbers. If we multiply $a - b$ by $a + b$, or $a + b$ by $a - b$, the product is $a^2 - b^2$, as is here shown.

$$\begin{array}{r} a - b \\ a + b \\ \hline a^2 - ab \\ ab - b^2 \\ \hline a^2 \quad - b^2 \end{array} \qquad \begin{array}{r} a + b \\ a - b \\ \hline a^2 + ab \\ -ab - b^2 \\ \hline a^2 \quad - b^2 \end{array}$$

Therefore, *the product of the sum and the difference of two numbers is the difference of their squares.*

That is, $$(a + b)(a - b) = a^2 - b^2.$$

Similarly, $$(2x + 1)(2x - 1) = 4x^2 - 1,$$

$$(5a + 3b)(5a - 3b) = 25a^2 - 9b^2,$$

$$33 \times 27 = (30 + 3)(30 - 3) = 900 - 9 = 891.$$

Exercise 60. Product of the Sum and Difference

Examples 1 to 23, oral — Examples 24 to 27, written

1. $(x + y)(x - y)$.
2. $(p + q)(p - q)$.
3. $(a + 7)(a - 7)$.
4. $(9 + m)(m - 9)$.
5. $(a + 6)(6 - a)$.
6. $(3x + 8)(8 - 3x)$.
7. $(p^4 + 3)(p^4 - 3)$.
8. $(2x^2 + 1)(2x^2 - 1)$.
9. $(3x^4 + 1)(3x^4 - 1)$.
10. $(5x^7 + 1)(1 - 5x^7)$.
11. $(2x + 3y)(2x - 3y)$.
12. $(3a + 9b)(3a - 9b)$.
13. $(4a^2 + 2)(4a^2 - 2)$.
14. $(a^m b^m + 1)(a^m b^m - 1)$.
15. What is the product of $20 + 4$ and $20 - 4$? of 24 and 16?
16. 32×28.
17. 34×26.
18. 41×39.
19. 42×38.
20. 51×49.
21. 52×48.
22. 61×59.
23. 92×88.

Multiply as indicated:

24. $(12x^6 + 16)(12x^6 - 16)$.
25. $(15a^4x^4 + 35)(15a^4x^4 - 35)$.
26. $(36 + 17b^m)(36 - 17b^m)$.
27. $(29a^{17} + c^{17})(29a^{17} - c^{17})$.

89. Products of Binomials. If two binomials have a common term, their product can easily be written. For example, study the following:

$$\begin{array}{r} x+7 \\ x+5 \\ \hline x^2+7x \\ \quad 5x+35 \\ \hline x^2+12x+35 \end{array} \qquad \begin{array}{r} x+a \\ x+b \\ \hline x^2+ax \\ \quad bx \quad +ab \\ \hline x^2+(a+b)x+ab \end{array}$$

The product of two binomials having a common term equals the square of the common term, plus the product of the common term by the sum of the other terms, plus the product of the other terms.

That is, $(x+a)(x+b)=x^2+(a+b)x+ab.$

Thus $(x+7)(x-3)=x^2+4x-21,$

because $+7+(-3)=4$, and $+7\times(-3)=-21.$

Similarly, $(x-9)(x-6)=x^2-15x+54,$

$(a+7)(a+7)=a^2+14a+49$ (as in § 87),

and $(a+7)(a-7)=a^2+0-49$ (as in § 88).

Exercise 61. Products of Binomials

Examples 1 to 20, oral — Examples 21 to 30, written

1. $(a+2)(a+3)$.	11. $(a+2)(a-3)$.	21. $(a-2)(a-13)$.
2. $(a+5)(a+7)$.	12. $(a-5)(a+7)$.	22. $(a-5)(a-27)$.
3. $(x+9)(x+3)$.	13. $(x+9)(x-3)$.	23. $(x-9)(x+13)$.
4. $(a+4)(a+5)$.	14. $(a-4)(a+5)$.	24. $(a-4)(a-35)$.
5. $(a+7)(a+3)$.	15. $(a+7)(a-3)$.	25. $(a-7)(a+13)$.
6. $(p+9)(p+4)$.	16. $(p-9)(p+4)$.	26. $(p-9)(p-24)$.
7. $(x+6)(x+2)$.	17. $(x+6)(x-2)$.	27. $(x-6)(x+42)$.
8. $(x+5)(x+9)$.	18. $(x-5)(x+9)$.	28. $(x-5)(x-19)$.
9. $(x+2)(x+7)$.	19. $(x+2)(x-7)$.	29. $(x-2)(x+27)$.
10. $(x+8)(x+7)$.	20. $(x-8)(x+7)$.	30. $(x-8)(x-17)$.

90. The Cube of a Binomial. If we raise $a+b$ to the third power, we shall have $(a+b)(a+b)(a+b)$, which is the same as $(a+b)(a+b)^2$, or $(a+b)(a^2+2ab+b^2)$. Multiplying, we have

$$\begin{array}{r} a^2+2ab\ \ +b^2 \\ a\ \ +b \\ \hline a^3+2a^2b+\ \ ab^2 \\ \qquad\quad a^2b+2ab^2+b^3 \\ \hline a^3+3a^2b+3ab^2+b^3 \end{array}$$

The cube of the sum of two numbers is the cube of the first, plus three times the square of the first multiplied by the second, plus three times the first multiplied by the square of the second, plus the cube of the second.

That is, $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$,

and $(a-b)^3 = a^3-3a^2b+3ab^2-b^3$.

Similarly, $(a+2)^3 = a^3+3a^2\cdot 2+3a\cdot 2^2+2^3$
$= a^3+6a^2+12a+8$.

Exercise 62. The Cube of a Binomial

Examples 1 to 10, oral — Examples 11 to 21, written

1. $(x+y)^3$.	6. $(a+1)^3$.	11. $(a^2+1)^3$.
2. $(x-y)^3$.	7. $(a-1)^3$.	12. $(a^2-1)^3$.
3. $(p+q)^3$.	8. $(1-a)^3$.	13. $(1+a^2)^3$.
4. $(p-q)^3$.	9. $(1-x)^3$.	14. $(1-a^2)^3$.
5. $(m+n)^3$.	10. $(1-y)^3$.	15. $(1-x^2)^3$.

16. Cube $a+3$; $a-3$; $3+a$; $3-a$; a^2+3; a^2-3.

17. Cube $2a+b$; $2a-b$; $a+2b$; $a-2b$; $b-2a$.

18. Cube $3a+2b$; $3a-2b$; $2a+3b$; $2a-3b$; $3b-2a$.

19. Cube $4a+1$; $4a-1$; $1-4a$; $a+4$; $4+a$; $4-a$.

20. Cube $3a+7$. Evaluate for $a=10$.

21. Evaluate $4n+1$ and also its cube for $n=10$, and thus find the cube of 41.

91. Quotients from dividing Squares or the Difference of Squares.

Since $a^2 + 2ab + b^2 = (a + b)(a + b)$,

therefore $\dfrac{a^2 + 2ab + b^2}{a + b} = a + b.$

Since $a^2 - 2ab + b^2 = (a - b)(a - b)$,

therefore $\dfrac{a^2 - 2ab + b^2}{a - b} = a - b.$

Since $a^2 - b^2 = (a + b)(a - b)$,

therefore $\dfrac{a^2 - b^2}{a + b} = a - b,$

and $\dfrac{a^2 - b^2}{a - b} = a + b.$

Exercise 63. Special Quotients

Examples 1 to 12, oral — Examples 13 to 21, written

1. Divide $x^2 + 4x + 4$ by $x + 2$.
2. Divide $a^2 - 6a + 9$ by $a - 3$.
3. Divide $25x^2 - 4$ by $5x + 2$; by $5x - 2$.

Divide:

4. $p^2 - q^2$ by $p + q$.
5. $9x^4 - y^4$ by $3x^2 + y^2$.
6. $25a^4 - 1$ by $5a^2 - 1$.
7. $16 - x^4$ by $4 - x^2$.
8. $81 - m^4$ by $9 - m^2$.
9. $49p^4 - 64q^4$ by $7p^2 + 8q^2$.

State two binomials whose product is:

10. $4x^2 - 25y^2$.
11. $36p^2q^2 - 1$.
12. $49a^2b^2c^2 - 1$.
13. $(a + b)^2 - c^2$.
14. $a^2 - (b + c)^2$.
15. $a^2 - (b - c)^2$.

Divide:

16. $x^6 - 1$ by $x^3 + 1$.
17. $4a^6 - 9$ by $2a^3 - 3$.
18. $9p^6 - 25$ by $3p^3 + 5$.
19. $(x + y)^2 - z^2$ by $x + y - z$.
20. $(x - y)^2 - z^2$ by $x - y - z$.
21. $(x - y)^2 - z^2$ by $x - y + z$.

92. Dividing the Sum or Difference of Two Cubes. If we divide $a^3 + b^3$ by $a + b$, we obtain $a^2 - ab + b^2$. Therefore

The sum of the cubes of two numbers is divisible by the sum of the numbers, and the quotient is the square of the first, minus their product, plus the square of the second.

That is, $$\frac{a^3 + b^3}{a + b} = a^2 - ab + b^2.$$

If we divide $a^3 - b^3$ by $a - b$, we obtain $a^2 + ab + b^2$. Therefore

The difference of the cubes of two numbers is divisible by the difference of the numbers, and the quotient is the square of the first, plus their product, plus the square of the second.

That is, $$\frac{a^3 - b^3}{a - b} = a^2 + ab + b^2.$$

For example, $(m^3n^3 - p^6) \div (mn - p^2) = m^2n^2 + mnp^2 + p^4$.

Exercise 64. Dividing the Sum or Difference of Cubes

Examples 1 to 8, oral — Examples 9 to 22, written

1. $(x^3 + y^3) \div (x + y)$.
2. $(x^3 - y^3) \div (x - y)$.
3. $(p^3 + q^3) \div (p + q)$.
4. $(p^3 - q^3) \div (p - q)$.
5. $(m^3 + 1) \div (m + 1)$.
6. $(m^3 - 1) \div (m - 1)$.
7. $(1 - m^3) \div (1 - m)$.
8. $(x^3 + 8) \div (x + 2)$.
9. $(x^3y^3 - 8) \div (xy - 2)$.
10. $(a^3b^3 + c^3) \div (ab + c)$.
11. $(a^3b^3 - c^3) \div (ab - c)$.
12. $(c^3 - a^3b^3) \div (c - ab)$.
13. $(8a^3 + b^3) \div (2a + b)$.
14. $(8a^3 - b^3) \div (2a - b)$.
15. $(b^3 - 8a^3) \div (b - 2a)$.
16. $(27x^3 + 1) \div (3x + 1)$.

Divide as indicated:

17. $\dfrac{125a^3 + 8b^3}{5a + 2b}$.
18. $\dfrac{64a^3b^3 + 8c^3}{4ab + 2c}$.
19. $\dfrac{343m^3 + n^3}{7m + n}$.
20. $\dfrac{343m^3 - n^3}{7m - n}$.
21. $\dfrac{8x^3 - 125y^3}{2x - 5y}$.
22. $\dfrac{8x^3 - 64y^3z^3}{2x - 4yz}$.

93. Summary of Special Products and Quotients. The following special relations mentioned in this chapter should be memorized, both as formulas and in complete sentences:

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$
$$(a+b)(a-b) = a^2 - b^2$$
$$(x+a)(x+b) = x^2 + (a+b)x + ab$$
$$\frac{a^2+2ab+b^2}{a+b} = a+b$$
$$\frac{a^2-2ab+b^2}{a-b} = a-b$$
$$\frac{a^2-b^2}{a+b} = a-b, \text{ and } \frac{a^2-b^2}{a-b} = a+b$$
$$\frac{a^3+b^3}{a+b} = a^2-ab+b^2$$
$$\frac{a^3-b^3}{a-b} = a^2+ab+b^2$$

Exercise 65. Review of Special Products and Quotients

Examples 1 to 8, oral — Examples 9 to 17, written

1. $(a+m)^2$.
2. $(a-m)^2$.
3. $(a+m)^3$.
4. $(a-m)^3$.
5. $(2a+1)^2$.
6. $(2a-1)^2$.
7. $(1+2a)^2$.
8. $(1-2a)^2$.
9. $(p+3q)^3$.
10. $(p-5q)^3$.
11. $(3q-2p)^3$.
12. $(5q-1)^3$.
13. Multiply $x+7$ by $x+7$; by $9x-7$; by $7x+8$.

Divide as indicated:

14. $\dfrac{4m^2+4m+1}{2m+1}$.
15. $\dfrac{4m^2-4m+1}{2m-1}$.
16. $\dfrac{49a^2b^2+14ab+1}{7ab+1}$.
17. $\dfrac{49a^2b^2-14ab+1}{7ab-1}$.

CHAPTER X

FACTORS

94. Factor. Any one of two or more numbers which multiplied together form a product is called a *factor* of the product.

As already stated (§ 37), 2 and 3 are factors of 6; 2, π, and r are factors of $2\pi r$, and $(b + b')$ and h are factors of $(b + b')\,h$.

A factor containing a letter is called a *literal factor;* one that is expressed by a numeral is called a *numerical factor.* In the expression $2a(4 + c)$, 2 is a numerical factor; a and $4 + c$ are literal factors.

Except where the contrary is stated, factors are limited to expressions that do not contain fractions or indicated roots. For example, although $3 \cdot \frac{2}{3} = 2$, and $n \cdot \frac{a}{n} = a$, we do not speak of 3 and $\frac{2}{3}$ as factors of 2, nor of n and $\frac{a}{n}$ as factors of a. In the same way, although $(\sqrt{x} + \sqrt{a})(\sqrt{x} - \sqrt{a}) = x - a$ (§ 88), we would not, at this stage, speak of $x - a$ as factorable. If fractions occur as coefficients, they are admitted as factors. For example, $\frac{2}{3}a + \frac{2}{3}b = \frac{2}{3}(a + b)$.

95. Prime and Factorable Expressions. An expression that contains no factors except itself and unity is said to be *prime.*

An expression that contains factors other than itself and unity is said to be *factorable.*

In factoring an expression we separate it into its prime factors. Thus the factors of a^3 are given as a, a, and a, although a^2 is also a factor of a^3.

96. Reduction of Fractions. We use factoring in reducing fractions to lowest terms, exactly as in arithmetic.

Just as $\frac{6}{15}$ is reduced to $\frac{2}{5}$ by canceling the common factor 3, that is, by dividing both terms by 3, so $\frac{a^2}{ab}$ is reduced to $\frac{a}{b}$ by canceling the common factor a. The subject is more fully treated in Chapter XI.

97. Factors of Monomials. The factors of a monomial are found by inspection.

For example, the factors of $6a^2b^3$ are 6, a, a, b, b, b. In factoring a monomial we do not attempt to factor the numerical coefficient. Similarly, we speak of the factors of $6a + 6b$ as 6 and $a + b$, although the 6 is also factorable.

To reduce $\frac{4}{6}$ to lowest terms we divide both terms by 2, the fraction becoming $\frac{2}{3}$. This is called canceling the factor 2. Similarly, to reduce $\frac{a^2b}{ab^2}$ to lowest terms we divide both terms by (or cancel) the common factors a and b, that is, the factors that are in both a^2b and ab^2, the result being $\frac{a}{b}$.

Exercise 66. Factors of Monomials

Examples 1 to 9, oral — Examples 10 to 27, written

1. Find the factors of a^2b^3; of a^3b^2; of $3a^2x^2y^2$; of $5ax^3y^4$.

2. Find the common factors of a^3b^2 and a^2b^3; of x^4y^4 and x^5y^3.

3. Find the common factors of m^7n^2 and m^5n^6; of x^9y^3 and x^3y^9; of abc^2 and a^2bc; of ab^2cd and abc^2d.

Reduce to lowest terms by canceling the factors common to numerator and denominator:

4. $\frac{a}{ab}$.
5. $\frac{b}{ab}$.
6. $\frac{a^2}{ab}$.
7. $\frac{b^2}{ab}$.
8. $\frac{ab}{a^2b^2}$.
9. $\frac{a^2b}{a^3b^2}$.
10. $\frac{a^4b^8}{a^3b^4}$.
11. $\frac{a^5b^7}{a^9b^4}$.
12. $\frac{a^9b^6}{a^5b^8}$.
13. $\frac{x^3y^8}{x^2y^2}$.
14. $\frac{x^7y^5}{x^2y^8}$.
15. $\frac{x^9y^8}{x^4y^8}$.
16. $\frac{3a^2b}{9ab^2}$.
17. $\frac{6a^4b^2}{9a^2b^2}$.
18. $\frac{8x^7y^8}{12x^7y^4}$.
19. $\frac{8m^6n^6}{14m^2n^2}$.
20. $\frac{5m^9n^8}{15m^8n^9}$.
21. $\frac{8p^7q^8}{20pq}$.
22. $\frac{17p^7q^8}{34pqr}$.
23. $\frac{17a^2b^2c^2}{51abc}$.
24. $\frac{19abc}{57a^2b^2c^2}$.
25. $\frac{17a^3b^2c}{85ab^2c^3}$.
26. $\frac{21p^4q^2r^4}{49p^2q^4r^2}$.
27. $\frac{27x^7y^7z^7}{81xyz}$.

98. Monomial Factors of Polynomials. When the terms of a polynomial have a common monomial factor the expression may be factored by inspection.

Required to factor $4a^3 + 6a^2b$.

It is evident that 2 is a factor of both terms, and that a^2 is also a factor of both. We may therefore write the expression thus:

$$4a^3 + 6a^2b = 2a^2 \cdot 2a + 2a^2 \cdot 3b = 2a^2(2a + 3b).$$

While the factors of a^2 are a and a, it is customary to leave the result in the form given above.

We may check the work by assigning numerical values to the letters, as in multiplication and division. In the above case, if $a = 1$ and $b = 2$ we have $4 + 12 = 2(2 + 6)$.

Exercise 67. Monomial Factors of Polynomials

Examples 1 to 10, oral — Examples 11 to 22, written

1. Factor $ax + bx$; $ax + ay$; $ax + bx + cx$; $ax + ay + az$.
2. Factor $ax^2 - bx^2$; $a^2x - a^2y$; $a^2x + b^2x + c^2x + d^2x$.

Factor the following:

3. $5x^2 + 15x^3$.
4. $8x^2 - 12x^4$.
5. $6a^3 + 9a^2$.
6. $9a^2 - 6a^5$.
7. $a^3 + a^2 + a$.
8. $a^3 + 3a^2 - 4a$.
9. $x^4 + x^3y - x^2y^2$.
10. $a^5 - 3a^4 + 7a^3$.
11. $-x^4 - 3x^3 - 2x^2$.
12. $6a^2b - 9ab^2 + 3ab$.
13. $4p^3q + 6pq^3 - 8p^2q^2$.
14. $8x^4y^4 - 6x^2y^2 + 10x^3y^3$.
15. $5m^7n^2 - 10m^5n^3 + 15m^4n^4$.
16. $12abc + 8a^2b^2c^2 - 4a^3b^3c^3$.
17. $6a^5b^4c^3 - 4a^4b^3c^2 + 2a^3b^2c$.
18. $3x^3y^3z^3 + 6x^2y^2z^2 + 9xyz$.
19. $17x^6y^6 - 51x^4y^4 + 85x^2y^2$.
20. $-19m^2n^2 + 76m^3n^3 - 95m^4n^4$.

Factor the numerator and denominator. Then reduce to lowest terms by canceling all common factors from each:

21. $\dfrac{12a^3b^3c^3 - 15a^2b^2c^2}{15a^2b^2c^2 - 12abc}$.
22. $\dfrac{5x^2y^2 - 10x^3y^3 + 15x^4y^4}{5x^3y^3 - 5x^2y^2 - 10x^4y^4}$.

99. Polynomials factored by grouping Terms. If we do not find a monomial factor that is common to all the terms of a polynomial, we may frequently factor by grouping the terms as shown in the following examples:

1. Required to factor $ax + ay + bx + by$.

Factoring the first two terms and the last two terms, we have

$$ax + ay + bx + by = a(x + y) + b(x + y).$$

We now see that $x + y$ is a common factor of these two groups. Hence

$$a(x + y) + b(x + y) = (a + b)(x + y).$$

We may check the work by multiplying $x + y$ by $a + b$, the result being $ax + ay + bx + by$; or we may substitute 1 (or any other number) for the letters. If we substitute 1 for each letter in the factors, we have

$$(1 + 1)(1 + 1) = 2 \cdot 2 = 4,$$

and

$$ax + ay + bx + by = 1 + 1 + 1 + 1 = 4.$$

2. Required to factor $3ax + 4ay + 3bx + 4by$.

$$\begin{aligned} 3ax + 4ay + 3bx + 4by &= a(3x + 4y) + b(3x + 4y) \\ &= (a + b)(3x + 4y). \end{aligned}$$

3. Required to factor $ax + ay - bx - by$.

$$\begin{aligned} ax + ay - bx - by &= (ax + ay) - (bx + by) \\ &= a(x + y) - b(x + y) \\ &= (a - b)(x + y). \end{aligned}$$

4. Required to factor $a^2y + ab^2 - axy - b^2x$.

$$\begin{aligned} a^2y + ab^2 - axy - b^2x &= (a^2y + ab^2) - (axy + b^2x) \\ &= a(ay + b^2) - x(ay + b^2) \\ &= (a - x)(ay + b^2). \end{aligned}$$

5. Required to factor $a^2x + b^2x + a^2 + b^2$.

$$\begin{aligned} a^2x + b^2x + a^2 + b^2 &= x(a^2 + b^2) + (a^2 + b^2) \\ &= (x + 1)(a^2 + b^2). \end{aligned}$$

6. Required to factor $m^2x + m^2y + m^2z + x + y + z$.

$$\begin{aligned} m^2x + m^2y + m^2z + x + y + z &= m^2(x + y + z) + (x + y + z) \\ &= (m^2 + 1)(x + y + z). \end{aligned}$$

In all the above cases we may check by actual multiplication or by substituting 1 (or any other number) for the letters.

Exercise 68. Polynomials factored by grouping Terms

Examples 1 to 8, oral — Examples 9 to 34, written

1. Factor $x^4 + x^3$; $xy + y$; $ax + ay^2$; $ax + b^2x$.
2. Factor $p^3 + pq$; $pq + q^3$; $pq + p^3q^2$; $p^2q^2 - pq$.

Name a factor common to the first two terms, and another factor common to the last two terms, of the following:

3. $a^5 + 2a^2 + a^3b + 2b$.
4. $a^2b + ab^2 + 3a + 3b$.
5. $x^2y + xy^2 + xy + y^2$.
6. $m^3n^3 + m^2n^2 + 2mn + 2$.
7. $m^5n^2 - m^3n^5 + 7m^3 - 7n^3$.
8. $pqr + qrx + p^2 + px$.

Factor Exs. 3–8, and also the following:

9. $ax^2 + bx^2 + ay^2 + by^2$.
10. $ax^2 - bx^2 + ay^2 - by^2$.
11. $ax^2 + bx^2 - ay^2 - by^2$.
12. $ax^2 + ay^2 - bx^2 - by^2$.
13. $a^3 + a^2b + 3a + 3b$.
14. $a^4 + a^3b^3 + ab + b^4$.
15. $x^7 + x^4y + x^3y + y^2$.
16. $x^4 - x^3y + xy^2 - y^3$.
17. $p^3 + p^2q + pq^2 + q^3$.
18. $p^5 + pq^4 - p^4q - q^5$.
19. $6ab + 9a + 4b + 6$.
20. $a^3 + a^2b + a + b$.
21. $x^2 + ax + x + a$.
22. $x^2 - ax + x - a$.
23. $x^2y^2 + xy + xyz + z$.
24. $a^2 + am + an + mn$.
25. $a^3 + a^2b^2 + ab + b^3$.
26. $a^3b^3 + ab + a^2b^2 + 1$.

Factor the numerator and then reduce to lowest terms:

27. $\dfrac{a^2 + ab + 2a + 2b}{a + 2}$.
28. $\dfrac{m^2 + mn + m + n}{m + 1}$.
29. $\dfrac{a^2 + 2ab + a + 2b}{a + 1}$.
30. $\dfrac{ax^2 + bx^2 + ay + by}{x^2 + y}$.
31. $\dfrac{ab^2 + b^2 + 2a + 2}{b^2 + 2}$.
32. $\dfrac{a^3b^3 + a^2b^2 + ab + 1}{ab + 1}$.
33. $\dfrac{a^3m^3 - a^2m^2 + am - 1}{am - 1}$.
34. $\dfrac{a^2b^2c^2 - a^3b^3c^3 + 1 - abc}{1 - abc}$.

100. Factoring Trinomials that are Perfect Squares. It is evident from § 87 that we can easily factor a trinomial if it is a perfect square.

Extract the square root of the first and last terms, and connect these square roots by the sign of the middle term.

That is, $a^2 + 2ab + b^2 = (a + b)(a + b)$,
and $a^2 - 2ab + b^2 = (a - b)(a - b)$.

1. Factor $x^2 + 2x + 1$.

Since the first and last terms are respectively the squares of x and 1, and $2x$ is twice the product of x and 1, we have

$$x^2 + 2x + 1 = (x + 1)(x + 1).$$

2. Factor $x^6 - 2x^3y + y^2$.

$$x^6 - 2x^3y + y^2 = (x^3 - y)(x^3 - y).$$

Exercise 69. Factoring Trinomials that are Perfect Squares

Examples 1 to 11, oral — Examples 12 to 21, written

1. Find the square root of $a^2 + 2a + 1$; of $a^2 - 2a + 1$.
2. Find two equal factors of $a^4 + 2a^2 + 1$; of $a^2b^2 + 2ab + 1$

Factor the following orally:

3. $b^2 + 2bc + c^2$.
4. $b^2 - 2bc + c^2$.
5. $c^2 - 2cd + d^2$.
6. $d^2 + 2d + 1$.
7. $d^2 - 2d + 1$.
8. $d^2 + 4d + 4$.
9. $x^2 - 4x + 4$.
10. $x^2 - 6x + 9$.
11. $x^2 - 8x + 16$.

Write the factors of the following:

12. $49a^2 + 14ab + b^2$.
13. $49a^2 - 14ab + b^2$.
14. $81a^2 + 36ab + 4b^2$.
15. $81a^2 - 54ab + 9b^2$.
16. $121x^2 + 22xy + y^2$.
17. $121x^2 - 44xy + 4y^2$.
18. $121x^2 + 66xy + 9y^2$.
19. $121x^2 - 88xy + 16y^2$.
20. $49x^4y^4 + 42x^2y^2 + 9$.
21. $49x^4y^4 - 28x^2y^2 + 4$.

101. Factoring the Difference of Two Squares. The difference of the squares of two quantities is always factorable (§ 88).

The difference of the squares of two quantities is the product of the sum and difference of the quantities.

That is, $a^2 - b^2 = (a + b)(a - b)$.

Factor the binomial $x^8 - y^8$.

$$\begin{aligned} x^8 - y^8 &= (x^4 + y^4)(x^4 - y^4) \\ &= (x^4 + y^4)(x^2 + y^2)(x^2 - y^2) \\ &= (x^4 + y^4)(x^2 + y^2)(x + y)(x - y). \end{aligned}$$

Exercise 70. Factoring the Difference of Two Squares

Examples 1 to 8, oral — Examples 9 to 28, written

1. Factor $a^2 - b^2$; $x^2 - y^2$; $m^2 - n^2$; $p^2 - q^2$; $a^2 - 1$; $m^2 - 1$; $p^2 - 4$; $x^2 - 7^2$; $8^2 - x^2$; $8^2 - 7^2$.

2. Factor $a^2 - 9$; $b^2 - 16$; $c^2 - 25$; $d^2 - 36$; $p^2 - 49$; $q^2 - 64$; $64 - q^2$; $64 - a^2b^2$; $17^2 - x^2$; $x^2 - 13^2$; $17^2 - 13^2$.

3. Factor $25 - a^2$; $36 - b^2$; $49 - c^2$; $64 - d^2$; $81 - x^2$; $100 - y^2$; $100 - x^2y^2$; $x^2y^2 - 100$; $25^2 - p^2$; $q^2 - 15^2$; $25^2 - 15^2$.

Factor the following:

4. $a^2 - 4$.	9. $x^4y^4 - 1$.	14. $x^8y^8z^8 - a^8$.
5. $a^2 - 4b^2$.	10. $1 - x^4y^4$.	15. $x^8y^8 - a^8b^8$.
6. $4a^2 - b^2$.	11. $x^4 - 16$.	16. $1 - x^8y^8z^8$.
7. $a^2 - 9b^2$.	12. $16 - x^4$.	17. $169x^2 - 36$.
8. $9a^2 - b^2$.	13. $x^4 - 16y^4$.	18. $144x^2 - 0.25$.

Factor the following:

19. $(x + y)^2 - 4z^2$.	24. $x^2 - a^2 - b^2 - 2ab$.
20. $9(x - y)^2 - z^2$.	25. $a^2 - 6ab - 4c^2 + 9b^2$.
21. $(a - b)^2 - (c + d)^2$.	26. $x^4 + 4$. (Add $4x^2 - 4x^2$.)
22. $(a - b)^2 - (c - d)^2$.	27. $p^4 + p^2 + 1$.
23. $a^2 - 4c^2 + b^2 + 2ab$.	28. $4x^2 - 16y^2 + 1 + 4x$.

Exercise 71. Review and Applications

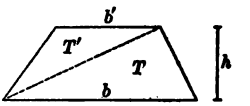
Examples 1 to 7, oral — Examples 8 to 29, written

1. Factor $ax - x^2$; $a^2 - ax$; $a^2 - x^2$.
2. Factor $a^2 + 2ab + b^2$; $a^2 + 6ab + 9b^2$; $a^2 - 6ab + 9b^2$.
3. Factor $4x^2 + 4x + 1$; $4x^2 - 4x + 1$; $x^2 - 4x + 4$.
4. Factor $a^2 - x^2$; $(a + b)^2 - x^2$; $(a + b)^2 - c^2$.
5. Factor $a^4 - x^4$; $m^4 - n^4$; $p^4 - q^4$; $b^4 - c^4$.
6. Factor $ax + bx + cx$; $ax - bx + cx$; $ax - bx - cx$.
7. Factor $x(a + b) + y(a + b)$; $x(a + b) + x(c + d)$.

Factor the following:

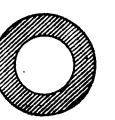
8. $a^4 - 2a^2b^2 + b^4$; $x^4 - 16y^4$; $x^8 - y^8$.
9. $4a^2 - 4a + 1$; $144x^2 - 24xy + y^2$.
10. $1 + 6x + 9x^2$; $4ac + 4c^2 + a^2 - 6x - 1 - 9x^2$.
11. $q^2 - 4qm^2 + 4m^4$; $p^4 - 6p^2r + 9r^2 - 4m^4 - q^2 + 4qm^2$.
12. $25x^2 - 9x^2m^2 - 1 - 10xy + y^2 + 6xm$.
13. $30p^3x + a^4x + 16p^4x - 25p^6x - 9x + 8a^2p^2x$.
14. $4ax^2 - 25ay^2 - ax^4 - 36ax + 81a + 10ax^2y$.

15. In the trapezoid here shown, the area of triangle T is $\frac{1}{2}bh$, and the area of triangle T' is $\frac{1}{2}b'h$. What is the area of the trapezoid? Factor the result, letting $\frac{1}{2}h$ be one factor.



16. In a cylinder the area of each base is πr^2, and the curved surface has an area of $2\pi rh$. This makes the total surface how much? Factor the result, thus simplifying the formula.

17. The area of the outside circle here shown is πa^2, and the area of the inside circle is πb^2. What is the area of the ring formed by the two circles? Factor the result, thus simplifying the formula.



Factor the following:

18. $4x^2 + 12xy + 9y^2 - 9a^2 - 24ab - 16b^2$.

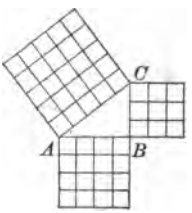
19. $x^2y^2z^2 + 2xyz + 1 - 2m^2n^2z - m^4n^4 - z^2$.

20. $1.21a^2b^2c^2 + 1 - x^2 - y^2 - 2.2abc - 2xy$.

21. $-2.64ab - x^2 + 1.44a^2 - y^4 + 1.21b^2 - 2xy^2$.

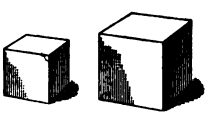
22. $-0.48ab - m^2 + 0.16a^2 - n^2 - 2mn + 0.36b^2$.

23. In the figure here shown, we have found that if $AB = b$, $BC = a$, and $AC = h$, then $h^2 = a^2 + b^2$, or $b^2 = h^2 - a^2$. Write the equation $b^2 = h^2 - a^2$ with the second member factored.



24. Using the factored form in Ex. 23, find the value of b^2 when $h = 5$, $a = 3$; when $h = 5$, $a = 4$; when $h = 35$, $a = 28$; when $h = 45$, $a = 36$.

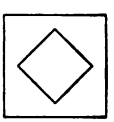
25. The edge of the larger of two cubes is a, and that of the smaller is b. The area of the base of one is how much greater than that of the other? Factor the result and evaluate for $a = 30$, $b = 20$.



26. In Ex. 25 what is the area of the entire surface of the six faces of each cube? What is the difference in area? Factor the result and evaluate for $a = 30$, $b = 20$.

27. The amount of principal and interest on a note is given by the formula $a = p + prt$. Factor the second member and evaluate for $p = 350$, $r = 6\%$, $t = 3$.

28. From a square of side a is cut a square of side b. What is the area of the remaining part? Factor the result and evaluate for $a = 60$, $b = 20$.



29. The area of the surface of a sphere of radius r is $4\pi r^2$. What is the difference of areas of two spheres of radii a and b, respectively? Factor the result and evaluate for $a = 5$, $b = 4$, and $\pi = 3\frac{1}{7}$.

102. Factoring the Quadratic Trinomial. A trinomial of the form $x^2 + bx + c$ is called a *quadratic trinomial.*

The more general form, $ax^2 + bx + c$, is considered in § 104.

To find the method of factoring this trinomial we consider the product of two binomials.

$$\begin{array}{r} x + 3 \\ x + 7 \\ \hline x^2 + 3x \\ \quad 7x + 21 \\ \hline x^2 + 10x + 21 \end{array} \qquad \begin{array}{r} x + a \\ x + b \\ \hline x^2 + \quad ax \\ \quad bx + ab \\ \hline x^2 + (a + b)x + ab \end{array}$$

We see that the factors of $x^2 + 10x + 21$ are $x + 3$ and $x + 7$, and that the factors of $x^2 + (a + b)x + ab$ are $x + a$ and $x + b$.

Hence, if a trinomial of the form $x^2 + px + q$ is factorable, the first term of each factor will be x; and the second terms of the factors will be two numbers whose product is q and whose algebraic sum is p, the coefficient of x.

Factor $x^2 + 5x - 36$.

Since the product of the second terms is -36, one must be positive and the other negative.

The algebraic sum of the two numbers is $+5$, hence the positive number must have the greater numerical value.

The two numbers whose product is -36 and whose algebraic sum is 5 are evidently 9 and -4.

Therefore $x^2 + 5x - 36 = (x + 9)(x - 4)$.

103. Directions for factoring $x^2 + bx + c$. In factoring an expression of the form $x^2 + bx + c$, we proceed as follows:

Find two monomials whose algebraic product is the absolute term with its proper sign, and whose algebraic sum is the coefficient of x with its proper sign.

Write for the factors two binomials, the first term of each being x, and the second terms being, respectively, the monomials thus found.

Exercise 72. Factoring the Quadratic Trinomial

Examples 1 to 10, oral — Examples 11 to 38, written

1. Find two numbers whose sum is 8 and whose product is 15; whose sum is 2 and whose product is -15.

2. Find two numbers whose sum is -2 and whose product is -15; whose sum is 2 and whose product is -35.

Find two numbers whose sum s and product p are:

3. $s = 8, p = 12$.
4. $s = 10, p = 24$.
5. $s = -5, p = -14$.
6. $s = -5, p = -6$.
7. $s = -1, p = -6$.
8. $s = -13, p = 40$.
9. $s = -16, p = 63$.
10. $s = -13, p = 22$.

Factor the following:

11. $a^2 + 8a + 15$.
12. $a^2 + 10a + 24$.
13. $x^2 - 5x - 14$.
14. $x^2 - 5x - 6$.
15. $p^2 - p - 6$.
16. $p^2 - 13p + 40$.
17. $n^2 - 16n + 63$.
18. $n^2 - 13n + 22$.
19. $m^2 - 2m - 15$.
20. $b^2 + 2b - 24$.
21. $b^2 + 5b - 14$.
22. $t^2 + 5t - 6$.
23. $x^2 + 5xy - 36y^2$.
24. $x^2 + xy - 12y^2$.
25. $x^2 - 4xy - 21y^2$.
26. $x^2 + xy - 72y^2$.
27. $x^2 + xy - 56y^2$.
28. $x^2 + 23x + 132$.
29. $x^2 + 3x - 130$.
30. $x^2 - 2x - 143$.
31. $x^2 + 5x - 150$.
32. $x^2 - 4x - 165$.
33. $x^2 + 30xy + 200y^2$.
34. $x^2 - 10xy - 200y^2$.

Factor both terms and then reduce to lowest terms:

35. $\dfrac{x^2 + 8x + 15}{x^2 + x - 20}$.
36. $\dfrac{a^2 + 10a + 24}{a^2 - a - 42}$.
37. $\dfrac{p^2 + 20p + 99}{p^2 + 21p + 108}$.
38. $\dfrac{x^2 + 22x + 121}{x^2 + 23x + 132}$.

104. Factoring the General Quadratic Trinomial. There is a more general form of the quadratic trinomial than that studied in § 102. For example, we may have the trinomial $6x^2 + 17x + 12$, or, more generally, $ax^2 + bx + c$.

There are several acceptable methods of factoring such expressions. The one requiring least explanation is given below; but if the teacher prefers, one of those given in the Appendix may be used instead.

If we consider the product of $3x + 4$ and $2x + 3$ we see that the first term of the product ($6x^2$) is the product of the first terms of the factors, $3x$ and $2x$; that the last term of the product (12) is the product of the last terms of the factors, 4 and 3; and that the middle term of the product ($17x$) is the sum of the products of the first term of either factor and the last term of the other factor, that is, the sum of their "cross products."

$$\begin{array}{r} 3x + 4 \\ 2x + 3 \\ \hline 6x^2 + 8x \\ \quad 9x + 12 \\ \hline 6x^2 + 17x + 12 \end{array}$$

Therefore, *in factoring an expression of the form $ax^2 + bx + c$, find two binomials such that the product of the first terms is the first term of the trinomial, and the product of the second terms is the last term of the trinomial, and such that the algebraic sum of their cross products is the middle term. These binomials are the required factors.*

As usual, the monomial factors are first to be removed.

Factor $6x^2 + 17x + 12$.

The factors of $6x^2$ are $3x$ and $2x$, or $6x$ and x.

The factors of 12 are 4 and 3, 6 and 2, or 12 and 1.

We must so select that the sum of the first factor of $6x^2$ and the last of 12, and the last factor of $6x^2$ and the first of 12, shall have as its algebraic sum $17x$.

We see that we cannot take $3x + 3$, because this contains the factor 3, which is not a factor of $6x^2 + 17x + 2$.

Similarly, we may reject $2x + 6$, $6x + 12$, $3x + 12$, $3x + 6$, and all other binomials that contain a monomial factor.

After rejecting these impossible factors we find, by a little trial, that $3x + 4$ and $2x + 3$ are the factors. We check by substituting some value for x, or by multiplication.

Exercise 73. Factoring the General Quadratic Trinomial

Examples 1 to 5, oral — Examples 6 to 41, written

1. In factoring $4x^2 + 8x + 3$, how will you proceed?

2. In factoring $64x^2 + 288x + 243$, how will you proceed?

3. In factoring $12x^2 - 19x - 21$, what do you know in advance as to the signs of the absolute terms of the factors?

4. In factoring $56x^2 - 22x + 2$, what do you know in advance as to the signs of the absolute terms of the factors?

5. After factoring $42x^2 - 85x + 42$, how will you check your work? Give two methods.

Factor the following:

6. $4x^2 + 8x + 3$.
7. $2x^2 + 5x + 3$.
8. $3x^2 - x - 2$.
9. $5x^2 - 8x + 3$.
10. $6x^2 + 7x + 2$.
11. $6x^2 - x - 2$.
12. $15x^2 + 14x - 8$.
13. $8x^2 - 10x + 3$.
14. $18x^2 + 9x - 2$.
15. $12x^2 - 5x - 2$.
16. $12x^2 - 7x + 1$.
17. $12x^2 - x - 1$.
18. $3x^2 - 2x - 5$.
19. $3x^2 + 4x - 4$.
20. $6x^2 + 5x - 4$.
21. $4x^2 + 13x + 3$.
22. $4x^2 + 11x - 3$.
23. $4x^2 - 4x - 3$.
24. $4x^2 + 16x + 7$.
25. $6x^2 + 17x + 12$.
26. $6x^2 + 17ax + 12a^2$.
27. $6a^2x^2 + 17axy + 12y^2$.
28. $6x^2y^2 + xy - 1$.
29. $12x^2 - 13xy - 14y^2$.
30. $10a^2 - 23ab - 5b^2$.
31. $8p^2 + 53pq - 21q^2$.
32. $8m^2 - 37mn - 15n^2$.
33. $2x^2 + 5xy + 2y^2$.
34. $8x^2 - 5bx - 3b^2$.
35. $8a^2 + 14ab - 15b^2$.
36. $60x^2 + 99x + 21$.
37. $45x^2 + 81x + 28$.
38. $125a^2b^2 + 135abc + 28c^2$.
39. $102a^2x^2 + 101axy - 21y^2$.
40. $130p^2q^2 + 21r^2 - 109pqr$.
41. $1 + 209x^2y^2z^2 - 30xyz$.

105. Factoring the Sum or Difference of Two Cubes. We have found (§ 92) that the sum of the cubes of two quantities is divisible by the sum of the quantities. Therefore, since

$$\frac{a^3+b^3}{a+b} = a^2 - ab + b^2,$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2).$$

Similarly, $$a^3 - b^3 = (a-b)(a^2 + ab + b^2).$$

Therefore, *the factors of the sum of the cubes of two quantities are* (1) *the sum of the quantities;* (2) *the square of the first, minus the product of the first and second, plus the square of the second.*

The factors of the difference of the cubes of two quantities are (1) *the difference of the quantities;* (2) *the square of the first, plus the product of the first and second, plus the square of the second.*

1. Factor $27x^3 + y^3$.

$$27x^3 + y^3 = (3x)^3 + y^3$$
$$= (3x + y)(9x^2 - 3xy + y^2).$$

2. Factor $8x^3 + 27y^3$.

$$8x^3 + 27y^3 = (2x)^3 + (3y)^3$$
$$= (2x + 3y)(4x^2 - 6xy + 9y^2).$$

3. Factor $27a^3 - 8b^3$.

$$27a^3 - 8b^3 = (3a)^3 - (2b)^3$$
$$= (3a - 2b)(9a^2 + 6ab + 4b^2).$$

4. Factor $(a-b)^3 + c^3$.

$$(a-b)^3 + c^3 = (a - b + c)[(a-b)^2 - (a-b)c + c^2]$$
$$= (a - b + c)(a^2 - 2ab + b^2 - ac + bc + c^2).$$

5. Factor $p^9 + q^{12}$.

Since $p^9 = p^3p^3p^3$, therefore $p^9 = (p^3)^3$. Similarly, $q^{12} = (q^4)^3$.

Therefore, $p^9 + q^{12} = (p^3)^3 + (q^4)^3$

$$= (p^3 + q^4)[(p^3)^2 - p^3q^4 + (q^4)^2]$$
$$= (p^3 + q^4)(p^6 - p^3q^4 + q^8),$$

because $(p^3)^2 = p^3p^3 = p^6$, and $(q^4)^2 = q^4q^4 = q^8$ (§ 60).

Exercise 74. The Sum or Difference of Cubes

Examples 1 to 9, oral — Examples 10 to 41, written

1. Factor a^3+b^3; a^3+1; x^3+1; a^3+2^3; m^3+3^3; p^3+q^3.

2. Factor a^3-b^3; x^3-1; m^3-2^3; p^3-3^3; a^3-8; x^3-27.

3. Factor m^3+1; m^3-1; x^3y^3+1; x^3y^3-1; $1+x^3y^3$.

4. Factor $1+a^3$; $1+x^3$; $1+m^3$; $1+p^3$; $1+q^3$; $1+y^3$.

5. Factor $1-a^3$; $1-x^3$; $1-m^3$; $1-p^3$; $1-q^3$; $1-y^3$; m^3n^3-1; $1-m^3n^3$; $a^3b^3c^3-1$; $1-x^3y^3$.

Factor the following:

6. $8x^3+1$.	**11.** $8x^3-27$.	**16.** m^3+8n^3.
7. $1+8x^3$.	**12.** $27-8x^3$.	**17.** $27m^3-n^3$.
8. $27x^3+1$.	**13.** $27x^3-125$.	**18.** $64x^3-1$.
9. $1+27x^3$.	**14.** $125-27x^3$.	**19.** $1-125x^3$.
10. $27x^3+8$.	**15.** $27x^3-8a^3$.	**20.** $216+x^3$.

21. Represent x^6+y^9 as the sum of the cubes of the two quantities, x^2 and y^3, and factor the result.

22. Represent $x^{12}-1$ as the difference of two cubes and factor the result.

Factor the following:

23. x^9-y^6.	**27.** $1-27x^6$.	**31.** $x^{15}+y^9$.
24. $x^{12}+y^6$.	**28.** $8-27x^9$.	**32.** $x^{15}-y^{12}$.
25. $x^{12}-y^9$.	**29.** $x^{12}+8y^{12}$.	**33.** $8x^{15}+y^6$.
26. $8x^{12}+1$.	**30.** $x^{15}-8$.	**34.** $27x^{15}-1$.

35. Represent a^6b^6+1 as the sum of two cubes and factor the result.

Factor the following:

36. a^3b^3+512.	**39.** $729+(a+b)^3$.
37. $a^3b^3-512(x+y)^3$.	**40.** $(a+b)^3+(c+d)^3$.
38. $(a+b)^3+512(x+y)^3$.	**41.** $(x+y)^3-(a+b)^3$.

106. Factoring Perfect Cubes. We have found (§ 90) that

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$

These statements may be combined, thus:

$$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3.$$

When there is the double sign, $\pm$, in both members, the upper signs go together and the lower signs go together.

Therefore *the cube root of a polynomial in the form* $a^3 \pm 3a^2b + 3ab^2 \pm b^3$ *is of the form* $a \pm b$.

Factor $8x^3 + 12x^2 + 6x + 1$; that is, find its cube root.

Since this polynomial equals $(2x)^3 + 3(2x)^2 + 3(2x) + 1$, it equals $(2x+1)^3$. The factors are therefore $2x+1$, $2x+1$, and $2x+1$. The answer should be written in the form $(2x+1)^3$.

Exercise 75. Factoring Perfect Cubes

Examples 1 to 6, oral — Examples 7 to 21, written

1. Factor $x^3 + 3x^2y + 3xy^2 + y^3$; $x^3 - 3x^2y + 3xy^2 - y^3$.
2. Factor $a^3 + 3a^2 + 3a + 1$; $a^3 - 3a^2 + 3a - 1$.
3. Factor $1 - 3m + 3m^2 - m^3$; $1 + 3xy + 3x^2y^2 + x^3y^3$.

Factor the following:

4. $2^3 + 3 \cdot 2^2 + 3 \cdot 2 + 1$.
5. $3^3 + 3 \cdot 3^2 + 3 \cdot 3 + 1$.
6. $7^3 + 3 \cdot 7^2 + 3 \cdot 7 + 1$.
7. $p^3 + 6p^2 + 12p + 8$.
8. $x^3 + y^3 + 3xy(x+y)$.
9. $a^3 + 6a^2 + 12a + 8$.
10. $a^3 - 6a^2 + 12a - 8$.
11. $8a^3 + 12a^2 + 6a + 1$.
12. $8a^3 - 12a^2 + 6a - 1$.
13. $8a^3b^3 + 12a^2b^2 + 6ab + 1$.
14. $1 - 6ab + 12a^2b^2 - 8a^3b^3$.
15. $8a^3b^3 + 12a^2b^2c + 6abc^2 + c^3$.
16. $8a^3b^3 - 12a^2b^2c + 6abc^2 - c^3$.
17. $27a^3 + 27a^2 + 9a + 1$.
18. $1 - 9a + 27a^2 - 27a^3$.
19. $64x^3 + 48x^2 + 12x + 1$.
20. $64x^3 + 12x - 48x^2 - 1$.
21. $x^3 - 12x^2y + 48xy^2 - 64y^3$.

107. Summary of Factoring. The student will find it of advantage to proceed as follows in factoring an expression:

First remove any monomial factor. Then see if the expression is one of the following forms already studied:

$ax + ay + bx + by.$	(§ 99)
$a^2 \pm 2ab + b^2.$	(§ 100)
$a^2 - b^2.$	(§ 101)
$x^2 + bx + c.$	(§ 102)
$ax^2 + bx + c.$	(§ 104)
$a^3 \pm b^3.$	(§ 105)
$a^3 \pm 3a^2b + 3ab^2 \pm b^3.$	(§ 106)

If so, factor as directed under these cases.

Factor each polynomial of the result until every factor is prime.

Students are urged to check their results in factoring, either by the substitution of some value for the letters or by multiplication, and teachers may well afford to insist upon it.

108. Changing the Signs of Factors. Since $abc = a \cdot -b \cdot -c = -a \cdot b \cdot -c = -a \cdot -b \cdot c$, we see that

The signs of any even number of factors may be changed without changing their product.

Therefore $a^2 - 2ab + b^2 = (a - b)(a - b)$ or $(b - a)(b - a)$,

and $a^2 - b^2 = (a + b)(a - b)$ or $-(a + b)(b - a)$.

Exercise 76. Review of Factoring

Examples 1 to 5, oral — Examples 6 to 63, written

1. Factor $ax + bx$; $ax - ay$; $am + am^2$; $ax + bx - cx$.
2. Factor $a^2 + 2a + 1$; $a^2 - 2a + 1$; $1 - 2a + a^2$.
3. Factor $m^2 - 1$; $1 - x^2$; $a^2b^2 - 1$; $a^2b^2 - c^2$.
4. Factor $a^3 + b^3$; $a^3 - b^3$; $a^3 + 1$; $1 + a^3$; $a^3 - 1$; $1 - a^3$.
5. Factor $a^3 + 3a^2 + 3a + 1$; $a^3 - 3a^2 + 3a - 1$.

Factor the following:

6. $x^4 - x^2$.
7. $x^4 - y^2$.
8. $x^4 - 1$.
9. $x^4 - y^4$.
10. $x^9 + y^9$.
11. $x^9 + 8y^9$.
12. $x^6 - y^6$.
13. $1 - y^6$.
14. $49a^2 - 16b^2$.
15. $49a^2 - (c + d)^2$.
16. $(a - b)^2 - 36c^2$.
17. $a^2b^2 - (c - d)^2$.
18. $1 - (c - d)^2$.
19. $(a + b)^3 + 1$.
20. $1 - (a - b)^3$.
21. $(x - y)^3 - 8$.
22. $x^2 + 14x + 49$.
23. $x^2 + 64 - 16x$.
24. $1 + 4a^2b^2 - 4ab$.
25. $a^2 - 11a + 18$.
26. $p^2 + 9p - 36$.
27. $4a^2b^2 + 12abcd + 9c^2d^2$
28. $9x^2y^2 + 4 - 12xy$.
29. $(a + b)^2 + 1 - 2(a + b)$.
30. $1 + (2x + y)^2 + 2(2x + y)$.
31. $m^2n^2 - 24mn + 95$.
32. $m^2n^4 + 24mn^2 + 95$.
33. $m^2n^6 + 95n^2 - 24mn^4$.
34. $a^2 + 2ab - c^2 + b^2$.
35. $a^2 - b^2 - c^2 + 2bc$.
36. $(p - q)^2 - (x - y)^2$.
37. $1 - x^2 - y^2 + 2xy$.
38. $1 - x^2 - 4y^2 - 4xy$.
39. $m^2 + 2mn + 3pm + 6np$
40. $9a^2 + 30ab + 25b^2$.
41. $9a^4 + 21a^2b^2 + 25b^4$.
42. $5a^2 + 13a - 6$.
43. $15a^2 + 19a + 6$.
44. $9a^2 + 9a + 2$.
45. $3p^2q^2 + 7pq - 6$.
46. $3p^2 + 7pq - 6q^2$.
47. $6a^2 + 7ab + 2b^2$.
48. $6a^2 - ab - b^2$.
49. $12a^2 - 7ab + b^2$.
50. $12a^2 + 5ab - 2b^2$.
51. $8a^2 + 2ab - b^2$.
52. $49x^4 - 81y^2$.
53. $a^2 + 2ab - 99b^2$.
54. $m^6n^6 + 8t^3$.
55. $49m^2n^2 - 121p^2$.
56. $121x^2y^2 - 36z^2$.
57. $x^2 - 20xy + 51y^2$.
58. $64a^2b^2 - 49c^4d^4$.
59. $729x^3 - y^3$
60. $729x^3 - 343$.
61. $x^3 - 5x^2 - 84x$.
62. $32x^2 + 4xy - 15y^2$.
63. $81x^4 - 625y^4$.

CHAPTER XI

FRACTIONS

109. Algebraic Fraction. An expression in the form of $\frac{a}{b}$, in which either a or b is an algebraic expression, is called an *algebraic fraction.*

For example, $\frac{2}{x}$, $\frac{x+y}{x-y}$, and $\frac{x+y}{2}$ are algebraic fractions.

Since we cannot divide by zero, b cannot be zero.

Because of its relation to factoring we have already (§ 96) introduced some simple work in fractions.

110. Terms of a Fraction. In the fraction $\frac{a}{b}$, a is called the *numerator*, b is called the *denominator*, and the two together are called the *terms* of the fraction.

The numerator represents the dividend and the denominator represents the divisor of an indicated division, just as in arithmetic.

111. Reduction of Fractions. As already stated (§ 96), we reduce algebraic fractions to fractions having lower terms just as in arithmetic, by dividing both terms by their common factors. Similarly, we reduce to fractions having higher terms by multiplying both terms by the same quantity.

Just as $\frac{10}{15}=\frac{2}{3}$, so $\frac{a+b}{a^2-b^2}=\frac{a+b}{(a+b)(a-b)}=\frac{1}{a-b}$.

Similarly, $\frac{a}{b}=\frac{ax}{bx}$, and $\frac{a}{x}=\frac{a\cdot ax}{x\cdot ax}=\frac{a^2x}{ax^2}$.

Multiplying or dividing both numerator and denominator of a fraction by the same expression does not change the value of the fraction.

112. Reduction of Fractions to Lowest Terms. A fraction is reduced to a fraction in lowest terms when the latter fraction has its numerator and denominator prime to each other.

Briefly, this is called the reduction of a fraction to lowest terms.

To reduce a fraction to lowest terms, divide both numerator and denominator by their common factors.

We might first find the single factor of highest degree common to both numerator and denominator, and then cancel it, but this is rarely done. The subject is discussed in Book II.

When a line is drawn through the factors by which both terms of the fraction are divided, the factors are said to be *canceled.*

We have already considered this subject in § 96 and in connection with various cases in factoring.

1. Reduce $\frac{24\,x^3y^4z}{32\,x^2y^5z^3}$ to lowest terms.

Dividing both terms by 8, x^2, y^4, and z we have $\frac{3x}{4yz^2}$.

2. Reduce $\frac{a^3-b^3}{a^4-b^4}$ to lowest terms.

$$\frac{a^3-b^3}{a^4-b^4}=\frac{\cancel{(a-b)}(a^2+ab+b^2)}{\cancel{(a-b)}(a+b)(a^2+b^2)}=\frac{a^2+ab+b^2}{(a+b)(a^2+b^2)}.$$

3. Reduce $\frac{x^3+5x^2+8x+6}{x^3-2x-4}$ to lowest terms.

Since it is difficult to factor numerator and denominator we resort to a simple and convenient device.

Since a factor of each of two quantities is a factor of their difference (just as a is a factor of ab and ac, and hence of $ab-ac$), we subtract the denominator from the numerator. The result is $5x^2+10x+10$.

Hence if there is any common factor it is contained in $5x^2+10x+10$, or $5(x^2+2x+2)$. Evidently 5 is not a common factor. Trying x^2+2x+2 we find, by actual division, that it is a common factor. Therefore we have

$$\frac{x^3+5x^2+8x+6}{x^3-2x-4}=\frac{\cancel{(x+3)(x^2+2x+2)}}{\cancel{(x-2)(x^2+2x+2)}}=\frac{x+3}{x-2}.$$

Students should be warned against canceling *terms;* only *factors* can be canceled.

Exercise 77. Reduction of Fractions to Lowest Terms

Examples 1 to 3, oral — Examples 4 to 27, written

1. Reduce to lowest terms: $\frac{a}{ab}$, $\frac{a^3}{a^2b}$, $\frac{a^2}{a^3b}$, $\frac{a^2b^2}{a^3b}$.

2. Reduce to lowest terms: $\frac{ax}{ay}$, $\frac{a^2x}{ay^2}$, $\frac{2a^2x}{4ay^2}$, $\frac{5a^2x^2}{10a^3y^3}$.

3. Reduce to lowest terms: $\frac{abc}{a^2bc}$, $\frac{abc}{a^2b^2c^2}$, $\frac{abc^2}{a^2b^2c^2}$, $\frac{a^2b^2c^2}{a^3b^3c^3}$.

Reduce the following to lowest terms:

4. $\frac{8a^2b^3c^4}{12a^4b^3c^2}$.
5. $\frac{15a^4b^6c^8}{20a^8b^5c^{10}}$.
6. $\frac{24a^{10}b^8}{32a^8b^{10}}$.
7. $\frac{36a^{10}b^{10}}{48a^{12}b^{15}}$.
8. $\frac{35x^4y^4z}{49x^5y^8z}$.
9. $\frac{50x^8y^9z^2}{75x^{10}y^{10}z}$.
10. $\frac{56x^ny}{72x^ny^2}$.
11. $\frac{63x^ny^2}{81x^ny^{10}}$.

12. Reduce $\frac{(a+b)^2}{a^2-b^2}$ to lowest terms, and check the result by letting $a=3$, $b=2$.

Reduce the following to lowest terms:

13. $\frac{a^2-b^2}{(a-b)^2}$.
14. $\frac{a^2-b^2}{a^3-b^3}$.
15. $\frac{a^3-1}{a^2-1}$.
16. $\frac{m^4-n^4}{m^2+n^2}$.
17. $\frac{p^3+q^3}{p^6+q^6}$.
18. $\frac{x^4-16}{x^6-64}$.
19. $\frac{x^3-y^3}{x^6-y^6}$.
20. $\frac{x^2-y^2}{x^5-y^5}$.

21. Reduce $\frac{1+4a+4a^2}{1-4a^2}$ to lowest terms, and check the result by letting $a=2$.

Reduce the following to lowest terms:

22. $\frac{a^3+a^2b}{a^2+2ab+b^2}$.
23. $\frac{x^4-y^4}{x^2-2xy+y^2}$.
24. $\frac{4x^2-7x+3}{5x^2-3x-2}$.
25. $\frac{x^3+2x^2+2x+1}{x^3+3x^2+3x+2}$.
26. $\frac{x^3-8x-8}{x^3+x^2-10x-12}$.
27. $\frac{a^3-6a^2-6a+1}{a^3-2a^2-34a+5}$.

113. The Sign of a Fraction. The plus or minus sign before a fraction is called the *sign of the fraction.*

If there is no sign expressed, the plus sign is understood as usual.

114. Changing Signs in the Terms. Since, from the law of signs in division (§ 70),

$$\frac{a}{b} = \frac{-a}{-b} = -\frac{-a}{b} = -\frac{a}{-b},$$

we see that

The value of a fraction is not altered by changing the signs of the numerator and denominator; by changing the signs of the fraction and numerator; or by changing the signs of the fraction and denominator.

It must be remembered that to change the sign of the numerator means that we must change the sign of *every term* of the numerator, and similarly for the denominator. Failure to do this is the cause of many errors that arise in work with fractions.

115. Equivalent Fractions. When two fractions are such that either may be obtained from the other by multiplying or dividing both of its terms by the same expression, they are called *equivalent fractions.*

For example, $\frac{6}{12}$ and $\frac{1}{2}$ are equivalent fractions. Likewise $\frac{a+b}{a-b}$ and $\frac{(a+b)^2}{a^2-b^2}$ are equivalent fractions.

116. Common Denominator. Two or more fractions that have the same denominator are said to have a *common denominator.*

For example, $\frac{3}{8}$, $\frac{5}{8}$, $\frac{6}{8}$, and $\frac{12}{8}$ have a common denominator.

117. Lowest Common Denominator. If the common denominator of several fractions is of the lowest degree possible, the fractions are said to have the *lowest common denominator.*

For example, $\frac{(a+b)^2}{a^2-b^2}$ and $\frac{a^2+3ab+2b^2}{a^2-b^2}$ have a common denominator, a^2-b^2. But since they can be reduced to the equivalent fractions $\frac{a+b}{a-b}$ and $\frac{a+2b}{a-b}$, $a-b$ is their *lowest* common denominator (L.C.D.).

118. Finding the Lowest Common Denominator. When several fractions are given in their lowest terms, we reduce any one of them to an equivalent fraction having any required denominator by multiplying both terms by the same expression. Therefore any common denominator of several fractions must contain all the factors of the given denominators. Therefore

The L.C.D. of several fractions must contain all the factors of their denominators, and no other factors.

1. Reduce $\frac{a}{bc}$ and $\frac{b}{cd}$ to equivalent fractions having the lowest common denominator.

The L.C.D. must contain all the factors of bc and cd, and no other factors; that is, it must be of the lowest possible degree.

The L.C.D. must therefore contain b, c, and d, and no other factors, and hence is bcd.

Evidently if we multiply both terms of $\frac{a}{bc}$ by d, and both terms of $\frac{b}{cd}$ by b, each fraction will have the L.C.D. bcd.

Therefore the result is $\frac{ad}{bcd}$ and $\frac{b^2}{bcd}$.

Fractions should be expressed in lowest terms before beginning to find the L.C.D.

2. Reduce $\frac{x-y}{x^2+3xy+2y^2}$ and $\frac{x}{2x^2+3xy+y^2}$ to equivalent fractions having the lowest common denominator.

The L.C.D. must contain the two denominators, and must therefore contain all their factors.

$$x^2+3xy+2y^2=(x+y)(x+2y),$$

and $$2x^2+3xy+y^2=(x+y)(2x+y).$$

Therefore the $$\text{L.C.D.}=(x+y)(x+2y)(2x+y).$$

Dividing the L.C.D. by the given denominators we see that the multiplying factors are $2x+y$ and $x+2y$, respectively.

Therefore
$$\frac{x-y}{(x+y)(x+2y)}=\frac{(x-y)(2x+y)}{(x+y)(x+2y)(2x+y)}$$
$$\frac{x}{(x+y)(2x+y)}=\frac{x(x+2y)}{(x+y)(x+2y)(2x+y)}.$$

Exercise 78. Reducing Fractions to the Lowest Common Denominator

Examples 1 to 11, oral — Examples 12 to 27, written

1. Express with L.C.D.: $\frac{a}{b^2}, \frac{1}{ab}; \frac{1}{a^2}, \frac{1}{ab}; \frac{a}{b}, \frac{b}{c}$.

2. Express with L.C.D.: $\frac{1}{a-b}, \frac{1}{a+b}; \frac{1}{a+b}, \frac{1}{(a+b)^2}$.

Express the following with lowest common denominator:

3. $\frac{1}{a}, \frac{1}{b}$.

4. $\frac{1}{a^2}, \frac{1}{a^3}$.

5. $\frac{1}{ab^2}, \frac{1}{a^2b}$.

6. $\frac{a+b}{a-b}, \frac{1}{a+b}$.

7. $\frac{a-b}{a+b}, \frac{1}{a-b}$.

8. $\frac{ab}{xyz}, \frac{a^2b^2}{x^2y^2z^2}$.

9. $\frac{p}{3q^2}, \frac{q}{2p^2}$.

10. $\frac{3}{a+b}, \frac{2}{a-b}$.

11. $\frac{a}{x(y+z)}, \frac{b}{y(y+z)}$.

12. $\frac{a}{x+y}, \frac{b}{x-y}$.

13. $\frac{x}{x+3}, \frac{x}{x-3}$.

14. $\frac{a+b}{a-b}, \frac{a-b}{a^2+ab+b^2}$.

15. $\frac{p^2-1}{p^2+1}, \frac{p^2+1}{p^2-1}$.

16. $\frac{a-b}{a^3+b^3}, \frac{a+b}{a^2-ab+b^2}$.

17. $\frac{a^2+b^2}{a^2-b^2}, \frac{a+b}{a-b}$.

18. $\frac{1}{x^2-y^2}, \frac{1}{x^3-y^3}$.

19. $\frac{1}{a^3-b^3}, \frac{1}{b^2-a^2}$.

20. $\frac{1}{x+y}, \frac{2}{x-y}, \frac{3}{x^2-y^2}$.

21. $\frac{a^2}{a-b}, \frac{a}{a^2-b^2}, \frac{a-b}{a+b}$.

22. $\frac{1}{a-b}, \frac{1}{b-c}, \frac{1}{c-a}$.

23. $\frac{a+3}{a-3}, \frac{a^2+9}{a^2-9}, \frac{a^3+27}{a^3-27}$.

24. $\frac{x}{y+z}, \frac{y}{z+x}, \frac{z}{x+y}$.

25. $\frac{2}{a}, \frac{3}{b}, \frac{4}{c}, \frac{9}{a+b+c}$.

26. $\frac{1}{(a-b)(b-c)}, \frac{1}{(c-b)(a-c)}$.

27. $\frac{1}{(x-y)(y-z)}, \frac{1}{(x-z)(z-y)}$.

119. Addition and Subtraction of Fractions. In algebra, as in arithmetic, fractions are added by first expressing them with the lowest common denominator.

Thus we might add $\frac{3}{4}$ and $\frac{5}{8}$ by reducing them to $\frac{2}{8}\frac{4}{4}$ and $\frac{4}{8}\frac{0}{2}$, but it saves labor to reduce them to $\frac{6}{8}$ and $\frac{5}{8}$, the sum being $\frac{11}{8}$, or $1\frac{3}{8}$.

So we might add $\frac{a}{b}$ and $\frac{a^2}{b^2}$ by reducing them to $\frac{ab^2}{b^3}$ and $\frac{a^2b}{b^3}$ and then adding, but it saves labor to use the lowest common denominator and to add $\frac{ab}{b^2}$ and $\frac{a^2}{b^2}$, the sum being $\frac{ab+a^2}{b^2}$, or $\frac{a(a+b)}{b^2}$.

Find the algebraic sum of $\frac{a-b}{a+b}+\frac{a+b}{a-b}-\frac{a^2+b^2}{a^2-b^2}$.

Since to subtract a quantity is the same as to add its negative, we may treat of addition and subtraction at the same time.

The denominators are $a+b$, $a-b$, and $(a+b)(a-b)$.

Therefore the L.C.D. is $(a+b)(a-b)$, or a^2-b^2.

Reducing to equivalent fractions with this denominator, we have

$$\frac{(a-b)^2}{a^2-b^2}+\frac{(a+b)^2}{a^2-b^2}-\frac{a^2+b^2}{a^2-b^2}=\frac{a^2-2ab+b^2+a^2+2ab+b^2-a^2-b^2}{a^2-b^2}$$

$$=\frac{a^2+b^2}{a^2-b^2}.$$

It should be noticed that the fraction line is a symbol of aggregation, so that when the third fraction is subtracted the signs of both a^2 and b^2 become negative.

Check. Let $a=2$, $b=1$. We then have $\frac{1}{3}+\frac{3}{1}-\frac{5}{3}=\frac{1+9-5}{3}=\frac{5}{3}$. The result, $\frac{a^2+b^2}{a^2-b^2}$, also equals $\frac{5}{3}$.

In the above example the work may conveniently be arranged as follows:

The L.C.D is $(a+b)(a-b)$.

The respective multipliers are $a-b$, $a+b$, and 1.

$$\begin{aligned}(a-b)(a-b) &= \quad a^2-2ab+b^2 = \text{1st numerator.}\\ (a+b)(a+b) &= \quad a^2+2ab+b^2 = \text{2d numerator.}\\ -(a^2+b^2) &= -a^2 \qquad -b^2 = \text{3d numerator.}\\ \hline & \quad a^2 \qquad +b^2 = \text{sum of numerators.}\end{aligned}$$

Therefore the sum of the fractions $=\frac{a^2+b^2}{a^2-b^2}$.

120. Finding the Algebraic Sum of Fractions. Therefore, to find the algebraic sum of several fractions we proceed as follows:

If the fractions have the same denominators, write the algebraic sum of the numerators over the common denominator.

If the fractions have different denominators, express them as equivalent fractions having the lowest common denominator, and write the algebraic sum of the new numerators over the common denominator.

In either case, reduce the resulting fraction to lowest terms.

Exercise 79. Addition and Subtraction of Fractions

Examples 1 to 6, oral — Examples 7 to 45, written

1. Add $\frac{a}{b}, \frac{c}{b}$; $\frac{a}{x^2}, \frac{b}{x^2}$; also $\frac{a}{m}, -\frac{b}{m}$.

2. Add $\frac{a+b}{x}, \frac{a-b}{x}$; $\frac{a-b}{m}, \frac{a-b}{m}$; also $\frac{a+b}{c}, \frac{b+a}{c}$.

Add the following:

3. $\frac{a^2-b^2}{x+y}, \frac{a^2+b^2}{x+y}$.

4. $\frac{a^2-25}{x-y}, \frac{a^2+25}{x-y}$.

5. $\frac{a}{2}, \frac{b}{2}, -\frac{c}{2}, -\frac{b}{2}$.

6. $\frac{a-b}{4}, \frac{b-c}{4}, \frac{c}{4}$.

7. Add $\frac{a+b}{a-b}$ and $\frac{2ab}{a^2-b^2}$. Check the result by letting $a=3$, $b=2$.

8. $\frac{a+b}{4} - \frac{a+2b}{6}$.

9. $\frac{x-y}{2} + \frac{y-x}{12}$.

10. $\frac{2mn}{5} - \frac{m^2-2mn+n^2}{15}$.

11. $\frac{b-a}{5} + \frac{a-b}{3}$.

12. $\frac{x+y}{4} - \frac{x-y}{8}$.

13. $\frac{m^2-n^2}{5} + \frac{m^2+2mn+n^2}{10}$.

done

14. $\frac{a}{3}+\frac{3a}{4}-\frac{5a}{12}$.

15. $\frac{x}{5}-\frac{2x}{3}+\frac{7x}{15}$.

16. $\frac{2a}{7}-\frac{a}{3}+\frac{5a}{21}$.

17. $\frac{4m}{5}-\frac{7m}{10}+\frac{4m}{15}$.

18. $\frac{3}{a}+\frac{4}{a^2}-\frac{5}{a^3}$.

19. $\frac{3}{a^3}-\frac{2}{3a^2}+\frac{1}{a}$.

20. $\frac{2}{3y}+\frac{3}{4y^2}-\frac{5}{12}$.

21. $\frac{5}{4x}-\frac{7}{5x}+\frac{1}{10x^2}$.

22. $\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$.

23. $\frac{x}{y}-\frac{y}{z}+\frac{z}{x}$.

24. $\frac{x}{yz}+\frac{y}{zx}+\frac{z}{xy}$.

25. $\frac{x}{yz}-\frac{y}{zx}-\frac{z}{xy}$.

26. $\frac{a}{m^2}+\frac{b}{mn}-\frac{c}{n^2}$.

27. $\frac{m^2}{n^2}-\frac{2}{mn}+\frac{n^2}{m^2}$.

28. $\frac{x}{ab}+\frac{y}{bc}+\frac{z}{ca}$.

29. $\frac{a+b}{a-b}+\frac{a-b}{a+b}$.

30. $\frac{a+b}{a-b}-\frac{a+b}{a^3-b^3}$.

31. $\frac{x}{x-y}-\frac{y}{x^3-y^3}$.

32. $\frac{a^2}{(a-1)^2}-\frac{a}{a-1}$.

33. $\frac{3}{a-3}+\frac{4}{a+3}$.

34. $\frac{5}{a-2}-\frac{3}{a+2}$.

35. $\frac{x}{x^2-y^2}+\frac{y}{x^3-y^3}$.

36. $\frac{x}{x^2-y^2}-\frac{y}{x^3+y^3}$.

37. $\frac{a-b}{(a+b)^2}+\frac{a+b}{(a-b)^2}$

38. $\frac{x+5}{(x-5)^2}-\frac{x-5}{(x+5)^2}$.

39. $\frac{a+b}{a^3-b^3}+\frac{a-b}{a^3+b^3}$.

40. $\frac{x+7}{x-3}+\frac{x+3}{x-7}$.

41. $\frac{x^2+1}{x^2-4}-\frac{x^2-1}{x^2+4}$.

42. $\frac{a+5}{a+6}+\frac{a+6}{a+7}$.

43. $\frac{a^2+2ab+b^2}{a-b}+\frac{(a-b)^2}{a+b}$.

44. $\frac{a^2-2ab+b^2}{a+b}+\frac{(a+b)^2}{a-b}$.

45. $\frac{a^2-2ab+b^2}{a+b}+\frac{a^2+b^2}{a-b}$.

121. Mixed Expression. An algebraic expression that is the sum of an integer and a fraction is called a *mixed expression.*

For example, $2\frac{1}{3}$ is a mixed number, and $2a + \frac{1}{3}b$ is a mixed expression. In arithmetic the plus sign is omitted between the integer and fraction in a mixed number, $2\frac{1}{3}$ meaning $2 + \frac{1}{3}$. In algebra $a\,\frac{b}{c}$ means $a \times \frac{b}{c}$, and hence the plus sign must be used in a mixed expression, as in the case of $a + \frac{b}{c}$.

122. Reduction of a Fraction to a Mixed Expression. If the degree of the numerator of a fraction equals or exceeds that of the denominator, in a given letter, the fraction may be reduced to an integer or a mixed expression by dividing the numerator by the denominator.

Just as we reduce $\frac{2}{1}\frac{5}{5}$ to $1\frac{2}{3}$ by dividing, so we may reduce $\frac{a^2+b^2}{a+b}$ to $a - b + \frac{2b^2}{a+b}$. We cannot, however, reduce $\frac{x^2+y^2}{a+b}$, although the numerator is of the second degree and the denominator is of the first degree, because different letters are involved.

1. Reduce $\dfrac{4x^2 - 6xy + 3y^2}{2x - y}$ to a mixed expression.

Dividing, $\dfrac{4x^2 - 6xy + 3y^2}{2x - y} = 2x - 2y + \dfrac{y^2}{2x - y}$.

2. Reduce $\dfrac{a^3 + 3a^2b - 6ab^2 + 2b^3}{a^2 + 2ab + b^2}$ to a mixed expression.

Dividing, $\dfrac{a^3 + 3a^2b - 6ab^2 + 2b^3}{a^2 + 2ab + b^2} = a + b$, with a remainder of $-9ab^2 + b^3$. We may therefore write the result

$$a + b + \frac{-9ab^2 + b^3}{a^2 + 2ab + b^2},$$

or (§ 114)

$$a + b - \frac{9ab^2 - b^3}{a^2 + 2ab + b^2},$$

the latter being the more common form.

The work may be checked by substituting any value for the letters that will not make the denominator zero.

Exercise 80. Reduction of a Fraction to a Mixed Expression

Examples 1 to 11, oral — Examples 12 to 26, written

1. Reduce to an integer: $\frac{10}{2}$; $\frac{10\,a^2}{2}$; $\frac{10\,a^2}{2\,a}$; $\frac{a^2b}{ab}$; $\frac{(a+b)^2}{a+b}$.

2. Reduce to an integer: $\frac{a^2-b^2}{a-b}$; $\frac{a^2-b^2}{a+b}$; $\frac{a^3+b^3}{a+b}$; $\frac{a^3-b^3}{a-b}$.

3. Reduce to a mixed expression: $\frac{x+y}{x}$; $\frac{x-y}{x}$; $\frac{ab+c}{a}$.

Reduce to integral or mixed expressions:

4. $\frac{ax-ay}{a}$.
5. $\frac{x^4-y^4}{x^2+y^2}$.
6. $\frac{abx+aby}{ab}$.
7. $\frac{abx+1}{ab}$.
8. $\frac{mn^2+3}{mn}$.
9. $\frac{abc+c}{abc}$.
10. $\frac{m^6-n^6}{m^3-n^3}$.
11. $\frac{a+b}{a-b}$.

12. Reduce $\frac{a^2+3\,ab+4\,b^2}{a+b}$ to a mixed expression, and check the result by letting $a=2$ and $b=3$.

Reduce the following to integral or mixed expressions:

13. $\frac{x^2+3\,xy+y^2}{x+y}$.
14. $\frac{x^2-3\,xy-y^2}{x-y}$.
15. $\frac{x^2+15\,x+56}{x+7}$.
16. $\frac{x^2-15\,x+56}{x-8}$.
17. $\frac{a^2b^2+3\,ab+2}{ab+2}$.
18. $\frac{p^2+4\,pq+q^2}{p+q}$.
19. $\frac{m^2-13\,mn+4\,n^2}{m-n}$.
20. $\frac{m^3+3\,m^2n+3\,mn^2+n^3}{m^2+n^2+2\,mn}$.
21. $\frac{m^3+3\,m^2n+4\,mn^2+5\,n^3}{m(m+2\,n)+n^2}$.
22. $\frac{a^3+a^2b+ab^2+b^3}{a+b}$.
23. $\frac{a^3+b^3+c^3-3\,abc}{a+b+c}$.
24. $\frac{x^3+3\,x^2y-xy^2-y^3}{x+y}$.
25. $\frac{x^4+x^2y^2+y^4}{x^2+xy+y^2}$.
26. $\frac{x^4+4\,x^2y^2+16\,y^4}{x^2+2\,xy+4\,y^2}$.

123. Reduction of a Mixed Expression to a Fraction. Since the value of an expression is not changed if it is both multiplied and divided by the same expression, then $a = \frac{ab}{b}$, and $a + \frac{a}{b} = \frac{ab + a}{b}$. Therefore

To reduce a mixed expression to a fraction, multiply the integral part by the denominator, to the product add the numerator, and under the result write the denominator.

This is simply the same as adding two fractions. For example, $a + \frac{c}{d} = \frac{ad}{d} + \frac{c}{d} = \frac{ad + c}{d}$.

Reduce $a + 3b - \frac{a^2 - 2ab + 4b^2}{a - 3b}$ to a fraction.

Reducing $a + 3b$ to a fraction with $a - 3b$ for its denominator, we have

$$\frac{a^2 - 9b^2}{a - 3b} - \frac{a^2 - 2ab + 4b^2}{a - 3b} = \frac{a^2 - 9b^2 - a^2 + 2ab - 4b^2}{a - 3b} = \frac{2ab - 13b^2}{a - 3b}.$$

Exercise 81. Reduction of a Mixed Expression to a Fraction

Examples 1 to 3, oral — Examples 4 to 9, written

1. Reduce to an improper fraction: $2\frac{3}{8}$; $4\frac{5}{8}$; $9\frac{7}{8}$.

2. Reduce to a fraction: $a + \frac{b}{c}$; $a - \frac{b}{c}$; $a + b + \frac{c}{d}$.

3. Reduce to a fraction: $a + \frac{ac}{b + c}$; $a - \frac{ac}{b + c}$; $a + b + \frac{c}{a - b}$

Reduce the following to fractions:

4. $a + b + \frac{a^2 + b^2}{a - b}$.

5. $x + 2y + \frac{4y^2}{x - 2y}$.

6. $a^2 + ab + b^2 + \frac{2b^3}{a - b}$.

7. $\frac{a^2 + b^2}{a - b} + b + 4a$.

8. $a^2 - ab + b^2 - \frac{b^3}{a + b}$.

9. $\frac{x^2 + 2xy - y^2}{x + y} + x + y$.

124. Multiplication of Fractions. We multiply fractions in algebra in the same manner as in arithmetic. Just as $5 \times \frac{2}{3} = \frac{10}{3}$, so $a \cdot \frac{b}{c} = \frac{ab}{c}$. Therefore

To multiply a fraction by an integral expression, multiply the numerator, writing the product over the denominator.

Similarly, just as $\frac{2}{3}$ of $\frac{4}{5}$ equals $\frac{2 \times 4}{3 \times 5}$, or $\frac{8}{15}$, so $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$. Therefore

To multiply a fraction by a fraction, multiply the numerators together for a new numerator, and the denominators together for a new denominator.

Cancel factors common to any numerator and any denominator before multiplying.

1. Find the product of $\frac{x^2y}{2m^2n} \times 4 \times \frac{5mn^3}{6xy^4} \times \frac{3y^2}{5m}$

Indicating the multiplication, we have

$$\frac{3 \cdot 4 \cdot 5 \cdot mn^3x^2y^3}{2 \cdot 5 \cdot 6 \cdot m^3nxy^4}.$$

Canceling common factors, we have $\frac{n^2x}{m^2y}$.

2. Find the product of $\frac{x^2-x-6}{x^2+x-6} \cdot \frac{x^2+2x-8}{x^2-2x-8} \cdot \frac{x^2-x-12}{x^2+x-12}$.

Factoring, for ease in canceling, and then canceling common factors, we have

$$\frac{\cancel{(x+2)}\cancel{(x-3)}}{\cancel{(x-2)}\cancel{(x+3)}} \cdot \frac{\cancel{(x-2)}\cancel{(x+4)}}{\cancel{(x+2)}\cancel{(x-4)}} \cdot \frac{\cancel{(x+3)}\cancel{(x-4)}}{\cancel{(x-3)}\cancel{(x+4)}} = 1.$$

3. Simplify $\left(x-1-\frac{4}{x-1}\right)\left(x^3+3x^2+8x+24+\frac{72}{x-3}\right)$.

Reducing to fractional forms,

$$\frac{x^2-2x-3}{x-1} \cdot \frac{x^4-x^2}{x-3} = \frac{\cancel{(x-3)}(x+1)}{\cancel{x-1}} \cdot \frac{x^2(x+1)\cancel{(x-1)}}{\cancel{x-3}} = x^2(x+1)^2.$$

Check. Let $x=2$. Then $(2-1-4)(8+12+16+24-72) = -3 \cdot -12 = 36$, and $4 \cdot 9 = 36$.

Exercise 82. Multiplication of Fractions

Examples 1 to 12, oral — Examples 13 to 30, written

1. $b \times \frac{a}{b}$.
2. $4 \times \frac{a}{4b}$.
3. $4b \times \frac{a}{4b}$.
4. $ab \times \frac{a}{b^3}$.
5. $mn \cdot \frac{a}{mn}$.
6. $p^2q \cdot \frac{1}{pq}$.
7. $x^2y^2 \cdot \frac{1}{xy}$.
8. $x^3y \cdot \frac{1}{xy^3}$.
9. $\frac{a}{b} \cdot \frac{p}{q}$.
10. $\frac{m}{n} \cdot \frac{n}{m}$.
11. $\frac{m}{n^2} \cdot \frac{n}{m^2}$.
12. $\frac{ab}{xy} \cdot \frac{ax}{by}$.

13. Find the product of $\frac{2ax^2}{3by^2} \cdot \frac{3cz^2}{4ax^2} \cdot \frac{4by^2}{5cz^2}$. Check the result by letting $a = b = c = 1$, and $x = y = z = 2$.

14. $\frac{ab^2}{c^2d} \cdot \frac{bc^2}{d^2a} \cdot \frac{cd^2}{a^2b}$.
15. $\frac{a^2x}{b^2y} \cdot \frac{b^2y}{c^2z} \cdot \frac{c^2z}{a^2x}$.
16. $\frac{m^2n}{pq^2} \cdot \frac{mn^2}{p^2q} \cdot \frac{p^4q^4}{m^4n^4}$.
17. $\frac{(2a)^2}{b} \cdot \frac{(3b)^2}{c} \cdot \frac{(4c)^2}{a}$.
18. $\frac{-a}{b^2} \cdot \frac{-b}{c^2} \cdot \frac{-c}{a^2}$.
19. $\frac{(8ab)^3}{(3pq)^2} \cdot \frac{(6pq)^3}{(4ab)^2} \cdot \frac{abc}{pqr}$.

20. Find the product of $\frac{x^2-y^2}{x^2+y^2} \cdot \frac{x+y}{x-y} \cdot \frac{x^4-y^4}{4}$. Check the result by letting $x = 2$, $y = 1$.

21. $\frac{a^2-b^2}{a^2+b^2} \cdot \frac{4a}{a+b}$.
22. $\frac{a^2-b^2}{a^2+b^2} \cdot \frac{2a}{3b+3a}$.
23. $\frac{a-b}{a+b} \cdot \frac{a^2-b^2}{a^3-b^3}$.
24. $\frac{x-7}{x+3} \cdot \frac{9-x^2}{49-x^2}$.
25. $\frac{x^2-1}{x^2+1} \cdot \frac{1-x^4}{1+x^4}$.
26. $\frac{4x^3+8x^2}{x-3} \cdot \frac{9x-x^3}{2+x}$.
27. $\frac{(a+b)^2}{a-b} \cdot \frac{a^2-b^2}{a^2+b^2}$.
28. $\frac{(m-n)^2}{m+n} \cdot \frac{m^3+n^3}{n-m}$.
29. $\frac{x^2+xy}{x^2-2xy+y^2} \cdot \frac{y^2-x^2}{4x}$.
30. $\frac{4x}{3x-2} \cdot \frac{4-9x^2}{8x}$.

125. Reciprocal. If the product of two quantities is 1, each is called the *reciprocal* of the other.

Since $\frac{a}{b} \cdot \frac{b}{a} = 1$, the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$, and conversely.

126. Division of Fractions. Consider the following cases:

$\$2 \div \$3 = \frac{2}{3}$, the denomination "dollars" disappearing in the quotient.

2 ft. $\div$ 3 ft. $= \frac{2}{3}$, the denomination "feet" disappearing.

$\frac{2}{5} \div \frac{3}{5} = \frac{2}{3}$, the denomination "fifths" disappearing.

To divide $\frac{a}{b}$ by $\frac{c}{d}$ is the same as to divide $\frac{ad}{bd}$ by $\frac{bc}{bd}$ (§ 111).

$\frac{ad}{bd} \div \frac{bc}{bd} = \frac{ad}{bc}$, the denomination "$bd$ths" disappearing.

But $\frac{ad}{bc} = \frac{a}{b} \cdot \frac{d}{c}$. Therefore we get the same result in dividing $\frac{a}{b}$ by $\frac{c}{d}$ that we get by multiplying $\frac{a}{b}$ by $\frac{d}{c}$. Therefore

To divide by a fraction, multiply by its reciprocal.

1. Divide $\frac{x^2 - x - 6}{x^2 + 3x - 4}$ by $\frac{x^2 + x - 12}{x^2 + x - 2}$.

Multiplying by the reciprocal of the divisor,

$$\frac{x^2 - x - 6}{x^2 + 3x - 4} \cdot \frac{x^2 + x - 2}{x^2 + x - 12} = \frac{(x+2)\cancel{(x-3)}}{(x+4)\cancel{(x-1)}} \cdot \frac{\cancel{(x-1)}(x+2)}{\cancel{(x-3)}(x+4)} = \frac{(x+2)^2}{(x+4)^2}.$$

2. Simplify $\frac{4x^2 - 1}{6x^2 + 13x + 6} \div \frac{2x^2 + 3x - 2}{2x^2 - 5x - 12} \div \frac{2x^2 - 7x - 4}{3x^2 + 8x + 4}$

Multiplying by the reciprocals of the divisors,

$$\frac{4x^2 - 1}{6x^2 + 13x + 6} \cdot \frac{2x^2 - 5x - 12}{2x^2 + 3x - 2} \cdot \frac{3x^2 + 8x + 4}{2x^2 - 7x - 4}$$

$$= \frac{\cancel{(2x+1)(2x-1)}}{\cancel{(3x+2)(2x+3)}} \cdot \frac{\cancel{(2x+3)(x-4)}}{\cancel{(2x-1)(x+2)}} \cdot \frac{\cancel{(x+2)(3x+2)}}{\cancel{(2x+1)(x-4)}} = 1.$$

Check. Substituting 1 for x,

$$\frac{3}{25} + \frac{3}{-15} \div \frac{-9}{15} = \frac{3 \cdot -15 \cdot 15}{25 \cdot 3 \cdot -9} = 1.$$

corrected

Exercise 83. Division of Fractions

Examples 1 to 11, oral — Examples 12 to 24, written

1. Divide $\frac{a}{2}$ by $\frac{a}{4}$; $\frac{a}{4}$ by $\frac{a}{2}$; $\frac{2}{a}$ by $\frac{4}{a}$; $\frac{4}{a}$ by $\frac{2}{a}$; $\frac{4}{a}$ by $\frac{a}{2}$.

2. Divide $\frac{a+b}{x}$ by $\frac{a-b}{x}$; $\frac{a+b}{2x}$ by $\frac{a-b}{2x}$; $\frac{a+b}{x+y}$ by $\frac{a-b}{x+y}$

3. $\frac{ab}{xy} \div \frac{cd}{xy}$.

4. $\frac{ab}{xy} \div \frac{bc}{xy}$.

5. $\frac{abc}{xy} \div \frac{bcd}{xy}$.

6. $\frac{2x}{y} \div \frac{3x}{y}$.

7. $\frac{2x}{y} \div \frac{3w}{y}$.

8. $\frac{2x}{y} \div \frac{3xy}{y^2}$.

9. $\frac{m}{n} \div \frac{n}{m}$.

10. $\frac{a}{b} \div \frac{m}{n}$.

11. $\frac{ab}{cd} \div \frac{cd}{ab}$.

12. Divide $\frac{15a^2b}{16c^2d}$ by $\frac{3ab^2}{4cd^2}$, and check the result by letting $a = 2$, $b = 3$, $c = 4$, $d = 5$.

13. $\frac{25abc}{32xyz} \div \frac{15a^2b^2c^2}{28x^2y^2z^2}$.

14. $\frac{17x^2yz}{24m^3n^2} \div \frac{51xy^2z^3}{75m^4n^5}$.

15. $\frac{35a^4b^5c^6}{27x^4y^5z^6} \div \frac{28a^3b^2c}{81x^2y^3z^4}$.

16. $\frac{25m^3n^4z}{33a^4b^5c^6} \div \frac{35mnz^4}{44a^3b^3c^3}$.

17. $\frac{a^3+b^3}{a-b} \div \frac{a^3-b^3}{a+b}$.

18. $\frac{a^3+b^3}{a-b} \div \frac{a+b}{a^3-b^3}$.

19. $\frac{a^4-b^4}{a^4+b^4} \div \frac{a^2-b^2}{a^2+b^2}$.

20. $\frac{m^2+n^2}{m^2-n^2} \div \frac{2m^2+2n^2}{m-n}$.

21. Divide $\frac{p^2-4q^2}{p^2+4pq}$ by $\frac{p^2-2pq}{pq+4q^2}$, and check the result by letting $p = 5$, $q = 2$.

22. $\frac{a^2+b^2-c^2+2ab}{c^2-a^2-b^2+2ab} \div \frac{a+b+c}{b+c-a}$.

23. $\frac{a^3+b^3+3ab(a+b)}{a^3-b^3-3ab(a-b)} \div \frac{a(a+2b)+b^2}{a(a-2b)+b^2}$.

24. $\left(\frac{1+m}{1-m} - \frac{1-n}{1+n}\right) \div \left[1 + \frac{(1+m)(1-n)}{(1-m)(1+n)}\right]$.

done

CHAPTER XII

SIMPLE FRACTIONAL EQUATIONS

127. Clearing of Fractions. Multiplying both members of an equation by such a quantity as shall leave no fractions in the equation is called *clearing the equation of fractions.*

If $\frac{1}{2}x = 5$,

by multiplying by 2, $x = 10$, an equation cleared of fractions.

If $\frac{x}{a} + \frac{x}{b} = ab$,

by multiplying by ab, $bx + ax = a^2b^2$, an equation cleared of fractions.

To clear an equation of fractions, multiply both members of the equation by the lowest common multiple of the denominators.

If a fraction is preceded by a minus sign, change the sign of every term in the numerator when the denominator is removed.

After clearing of fractions, solve the equation in the usual manner.

1. Solve $\frac{x}{2} + \frac{3}{a} - 4 = \frac{1}{a} - \frac{1}{4}$.

Multiplying by $4a$, the L.C.M. of the denominators,

$$2ax + 12 - 16a = 4 - a.$$

Subtracting $12 - 16a$, $2ax = -12 + 16a + 4 - a$.

Combining, $2ax = 15a - 8$.

Dividing by $2a$, $x = \frac{15a - 8}{2a}$.

2. Solve $\frac{x+1}{x-1} = \frac{x-1}{x-2}$.

Multiplying by the L.C.M., $x^2 - x - 2 = x^2 - 2x + 1$.

Therefore $x = 3$.

Exercise 84. Fractional Equations

Examples 1 to 4, oral — Examples 5 to 20, written

1. Clear of fractions: $\frac{x}{a}=b$; $\frac{x}{2a}=\frac{a}{2}$; $\frac{x}{3}+\frac{1}{4}=1$.

2. Solve $\frac{x}{3}+1=2$; $\frac{x}{4}-1=2$; $\frac{x}{5}-2=2$.

3. Solve $\frac{2x}{3}=4$; $\frac{3x}{4}=6$; $\frac{4x}{5}=8$; $\frac{5x}{7}=10$.

4. Solve $\frac{x+1}{2}=3$; $\frac{x-1}{2}=5$; $\frac{x+1}{3}=6$; $\frac{x-1}{5}=2$.

Solve the following equations:

5. $\frac{x+1}{x-1}=\frac{5}{3}$.

6. $\frac{x-1}{x+1}=\frac{3}{4}$.

7. $\frac{x+2}{x-2}=1\frac{3}{5}$.

8. $\frac{x-2}{x+2}=\frac{3}{7}$.

9. $\frac{x+3}{x-5}=9$.

10. $\frac{x-3}{x-7}=3$.

11. $\frac{x+1}{x-1}=\frac{2x-5}{2x-7}$.

12. $\frac{x+2}{x-3}=\frac{2x-5}{2x-10}$.

13. $\frac{x-5}{x+6}=\frac{3x-17}{3x-6}$.

14. $\frac{2x-1}{2x+1}=1-\frac{2}{3x}$.

15. $\frac{2x+3}{2x-5}=\frac{4x-3}{4x-11}$.

16. $\frac{5x+1}{5x-1}=\frac{2x+3}{2(x+1)}$.

17. The sum of the seventh and ninth parts of a certain number is 16. Find the number.

18. The sum of one half, one third, and one fourth of a certain number is 13. Find the number.

19. The sum of one half and three fourths of a certain number is two more than the number. Find the number.

20. There are two consecutive numbers, x and $x+1$, such that half of the first plus one third of the second equals 7. Find the numbers.

128. Combining Terms before Clearing. It often simplifies the work to combine certain terms before clearing an equation of fractions.

1. Solve $\frac{x+1}{2a}+\frac{2x}{3a}=\frac{1}{2a}+\frac{x}{3a}+5$.

Here it is apparently advisable to unite the fractions with the same denominators before clearing of fractions.

Subtracting $\frac{1}{2a}$ and $\frac{x}{3a}$,

$$\frac{x+1-1}{2a}+\frac{2x-x}{3a}=5,$$

or

$$\frac{x}{2a}+\frac{x}{3a}=5.$$

Combining,

$$\frac{5x}{6a}=5.$$

Dividing by 5,

$$\frac{x}{6a}=1.$$

Multiplying by $6a$,

$$x=6a.$$

2. Solve $\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-5}{x-6}-\frac{x-6}{x-7}$.

Combining the fractions in the first member and those in the second member, we have

$$\frac{(x-1)(x-3)-(x-2)^2}{(x-2)(x-3)}=\frac{(x-5)(x-7)-(x-6)^2}{(x-6)(x-7)}$$

Simplifying the numerators,

$$\frac{-1}{(x-2)(x-3)}=\frac{-1}{(x-6)(x-7)}.$$

Multiplying by -1, and then clearing of fractions,

$$(x-6)(x-7)=(x-2)(x-3).$$

Expanding,

$$x^2-13x+42=x^2-5x+6.$$

Therefore

$$8x=36,$$

and

$$x=4\tfrac{1}{2}.$$

We might also reduce to mixed fractions and then solve, thus:

$$1+\frac{1}{x-2}-1-\frac{1}{x-3}=1+\frac{1}{x-6}-1-\frac{1}{x-7}.$$

Exercise 85. Fractional Equations

Examples 1 to 3, oral — Examples 4 to 13, written

1. Solve $x + \frac{3}{2} - \frac{1}{2} = 2$; $x + \frac{4}{5} = 3 - \frac{1}{5}$.

2. Solve $x + \frac{2}{3} - \frac{1}{3} = 2\frac{1}{3}$; $x + \frac{3}{4} + \frac{1}{4} = 4$.

3. Solve $x + \frac{7}{8} - \frac{3}{8} = 5\frac{1}{2}$; $x + \frac{6}{8} + \frac{3}{8} = 17$.

Solve the following equations:

4. $\dfrac{3x-5}{5x-5} + \dfrac{5x-1}{7x-7} + \dfrac{x-4}{x-1} = 2.$

5. $\dfrac{8x+2}{x-2} - \dfrac{2x-1}{3x-6} + \dfrac{3x+2}{5x-10} + 5 = 15.$

6. $\dfrac{3x-1}{2x-6} + \dfrac{5x-7}{3x-9} + \dfrac{7x+1}{4x-12} + 14 = 25.$

7. $\dfrac{3x}{2x-2} + \dfrac{5}{6(x-1)} - \dfrac{15x-7}{9(x+1)} = \dfrac{17x-x^2}{6(x^2-1)} + \dfrac{71}{18(x^2-1)}.$

8. $\dfrac{4}{x+1} + \dfrac{x}{1-x} - \dfrac{1}{x-1} = \dfrac{x^2-3}{1-x^2}.$

Solve the following equations:

9. $\dfrac{4}{x+2} + \dfrac{7}{3+x} - \dfrac{37}{x(x+5)+6} = 0.$

10. $\dfrac{1}{x-1} - \dfrac{1}{x-2} = \dfrac{1}{x-3} - \dfrac{1}{x-4}.$

11. $\dfrac{1}{x+1} + \dfrac{2}{x-1} = \dfrac{4}{x+1}.$

12. One fifteenth of $6x + 7$, divided by x, equals the quotient of $2x - 2$ divided by $5x - 6$. Find the value of x.

13. One fifteenth of $6x + 1$, diminished by the quotient of $2x - 4$ divided by $7x - 16$, equals one fifth of $2x - 1$. Find the value of x.

129. Monomial and Polynomial Denominators. When an equation contains fractions some of which have monomial denominators, it is usually advisable to clear of these fractions first.

Solve $x^2+\frac{1}{2}+\frac{9x-7}{6(3x+11)}=\frac{3x^2+2}{3}$.

Multiplying by 6, $6x^2+3+\frac{9x-7}{3x+11}=6x^2+4$.

Subtracting $6x^2+3$, $\frac{9x-7}{3x+11}=1$.

Multiplying by $3x+11$, $9x-7=3x+11$.

Solving this equation, $x=3$.

Exercise 86. Monomial and Polynomial Denominators

Examples 1 and 2, oral — Examples 3 to 8, written

1. Solve $\frac{x}{2}+\frac{x}{3}=5$; $\frac{x}{2}-\frac{x}{3}=10$; $\frac{x}{2}-\frac{x}{4}=7$.

2. Solve $\frac{2x}{3}+\frac{x}{3}=17$; $\frac{3x}{4}-\frac{x}{2}=11$; $\frac{5x}{8}-\frac{x}{8}=12$.

Solve the following equations:

3. $\frac{10x+17}{18}+\frac{4-5x}{9}=\frac{2(6x+1)}{13x-16}$.

4. $\frac{2x}{3}-\frac{2x+4}{3}+\frac{7}{9}=\frac{13-7x}{3(2x+1)}$.

5. $\frac{2x}{5}+\frac{2(1-x)}{7x-6}+\frac{7}{15}=\frac{2x+1}{5}$.

6. $\frac{9x+5}{14}-\frac{7-8x}{2(3x+1)}=\frac{36x+15}{56}+\frac{41}{56}$.

7. $\frac{11x-13}{14}-\frac{22x-75}{28}=\frac{13x+7}{2(3x+7)}$.

8. $\frac{x}{x-1}-\frac{5}{2x-2}=\frac{8}{3x-3}+\frac{x}{1-x}+\frac{5}{18}$.

130. Literal Equations. Equations in which some or all of the known quantities are represented by letters are called *literal equations.*

Known quantities are usually represented by the first letters of the alphabet, and unknown numbers by the last letters of the alphabet.

Thus, in the equation $ax + b = cx + d$, x is supposed to represent the unknown quantity unless otherwise stated, and a, b, c, and d to represent known quantities.

We shall see that there are exceptions to this custom, but since they are always apparent from the example they will offer no difficulty.

1. Solve $(a + x)(a - x) = x(3a - x)$.

Here, as usual, unless the contrary is stated, x is supposed to be the unknown quantity.

Simplifying,	$a^2 - x^2 = 3ax - x^2.$
Subtracting $-x^2$,	$a^2 = 3ax.$
Dividing by $3a$,	$\frac{1}{3}a = x.$
Check.	$\frac{4}{3}a \cdot \frac{2}{3}a = \frac{1}{3}a \cdot \frac{8}{3}a = \frac{8}{9}a.$

In the solution we may leave the unknown quantity in the second member, as here, or we may transpose, and then change the signs by multiplying both sides by -1, having

$$3ax = a^2,$$
$$x = \tfrac{1}{3}a.$$

2. Solve $ax + b^2 = a^2 - bx$.

Subtracting b^2 and $-bx$,	$ax + bx = a^2 - b^2.$
Factoring,	$(a + b)x = (a + b)(a - b).$
Dividing by $a + b$,	$x = a - b.$

3. Solve $\dfrac{x}{a} + b = \dfrac{x}{b} + a$.

Multiplying by ab,	$bx + ab^2 = ax + a^2b.$
Subtracting ab^2 and ax,	$bx - ax = a^2b - ab^2.$
Factoring,	$(b - a)x = -ab(b - a).$
Dividing by $b - a$,	$x = -ab.$
Check.	$\dfrac{-ab}{a} + b = \dfrac{-ab}{b} + a,$
for	$-b + b = -a + a.$

Exercise 87. Literal Equations

Examples 1 to 5, oral — Examples 6 to 29, written

1. If $ax = b$, what is the value of x?

2. If $(a-1)x = a^2 - 1$, what is the value of x?

3. Solve $4abx = 16a^2b^2$; $5a^2b^2x = 25a^3b^3$; $17a^2x = 51a^4$.

4. Solve $ax = a$; $(a-b)x = a-b$; $(a-b)x = b-a$; $(a-b)x = a^2 - 2ab + b^2$.

5. Solve $\frac{x}{a} = a$; $\frac{x}{a} = b$; $\frac{x}{2ab} = 1$; $\frac{x}{a+b} = a-b$.

Solve the following equations:

6. $2ax + 1 = ax + 7$.

7. $2ax + a = ax + b$.

8. $2x - a^2 = x + a^2$.

9. $x^2 + a^2 = (x+b)^2$.

10. $ax - a^2 = ab - bx$.

11. $a(a-x) = b(b-x)$.

12. $x^2 + ax = (x+a)^2$.

13. $\frac{c}{a(x+1)} = \frac{b}{c(1-x)}$.

14. $\frac{a+dx}{cd+d^2x} = \frac{a-x}{a-dx}$.

15. $\frac{a+bx}{a+b} = \frac{c+dx}{c+d}$.

16. $\frac{\frac{m+n}{x}}{\frac{1}{m}} = \frac{a}{b}$.

17. $2a^2bx - c = 2b^2cx - a$.

18. $4abcx - bc = 4bcx + 1$.

19. $(a+x)(x+b) = x(x-b)$.

20. $(a+x)(a-x) = x(b-x)$.

21. $(x+a)^2 = x(x-b)$.

22. $(x+a)(x-b) = x^2 + abc$.

23. $(4x+b)(x-b) = 4x(x+b)$.

24. $\frac{x-a}{x-b} = \left(\frac{2x-a}{2x-b}\right)^2$.

25. $\frac{a+b}{x+2} = \frac{b-a}{x-2}$.

26. $\frac{6x+a}{4x+b} = \frac{3x-b}{2x-a}$.

27. $\frac{\frac{ax-b}{2}}{\frac{a}{b}} = \frac{\frac{ax+b}{2}}{\frac{b}{a}}$.

28. $\frac{x+a}{b} - x = b - \frac{x-b}{c} + \frac{c-bx}{b}$.

29. $\frac{x^2+2ax+a^2}{a+x} + \frac{x^2+a^2-2ax}{x-a} = \frac{x-a}{b} + \frac{x-b}{a}$.

131. Letters used in Formulas. In writing formulas such as are used in measuring, physics, and business, it is convenient to use other letters than $a, b, c, \cdots$ for the known numbers, and x, y, z, for the unknown quantities. The initial letter of some word is often used for either a known or an unknown quantity.

For example, we say that $c = 2\pi r$, meaning that the circumference equals 2π times the radius. If we know what r equals and wish to find c, then we think of r as known and c as unknown. If we know what c equals and wish to find r, then c is known and r is unknown.

1. Given $c = 2\pi r$, to find r.

Dividing by 2π, that is, by $2 \times 3\frac{1}{7}$ or 2×3.1416,

$$\frac{c}{2\pi} = r.$$

2. Given $v = \pi r^2 h$, to find h.

Dividing by πr^2, $$\frac{v}{\pi r^2} = h.$$

3. Given $v = \pi r^2 h$, to find r.

Dividing by πh, $$\frac{v}{\pi h} = r^2.$$

Extracting the square root $$\sqrt{\frac{v}{\pi h}} = r.$$

132. Use of Primes and Subscripts. If we wish to use a formula in which two different radii occur, we have found that we may represent one radius by r and the other by r' (read "r prime").

For example, $c = 2\pi r$, and $c' = 2\pi r'$; then $c - c' = 2\pi(r - r')$.

Here we have the advantage of using the initial letter c for two different circumferences and the initial letter r for two different radii.

Sometimes it is convenient to use r' and r'' ("r second") or, preferably, r_1 and r_2 (read "r sub-one" and "r sub-two"). It is also customary, particularly in machine work, to use capital letters in formulas. We read r_1^2 "r sub-one square" or "the square of r sub-one." As already stated, we may distinguish R as "r major," and r as "r minor."

Exercise 88. Formulas as Equations

Examples 1 to 6, oral — Examples 7 to 38, written

1. Solve $a = \pi r^2$ for r.

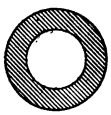
2. Solve $a = \frac{1}{2}bh$ for h; also for b.

3. Solve $v = bwl$ for b; also for w; also for l.

4. Solve $a = \frac{1}{2}h(b + b')$ for h. How will you solve for b?

5. Solve $i = prt$ for p; also for r; also for t.

6. Solve $a = p(1 + rt)$ for p. How will you proceed to solve for r?



7. The area of a ring is indicated by the formula $a = \pi(r_1 + r_2)(r_1 - r_2)$. Find the value of r_1.

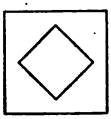
8. In Ex. 7, suppose $a = 286$, $r_2 = 10$, and $\pi = 3\frac{1}{7}$, find the value of r_1.

9. In Ex. 7, suppose $a = 506$, $r_2 = 12$, and $\pi = 3\frac{1}{7}$, find the value of r_1.

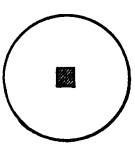
10. In Ex. 7, suppose $a = 528$ and $r_1 = 2r_2$, find the value of r_1 and r_2.

11. In connecting two wheels by a belt it is often required to know the difference in the two circumferences. This is given by the formula $D = 2\pi(r_1 - r_2)$. Suppose D and r_2 are known, find r_1.

12. If from a square of side s_1 there is cut out a smaller square of side s_2, the area of the part that remains is given by the formula $a = (s_1 + s_2)(s_1 - s_2)$. Find a when $s_1 = 11$, $s_2 = 9$; also find s_2 when a and s_1 are given.



13. If from a circle of radius r there is cut a square of side s, the area that remains is given by the formula $a = \pi r^2 - s^2$. If $r = 7$ and $s = 4$, find a. If a and s are known, find r. If a and r are known, find s.



The following formulas will be met by the student in his subsequent work in mathematics or physics. Solve as directed:

14. $s = vt$. Find t.

15. $s = vt$. Find v.

16. $s = \frac{1}{2}gt^2$. Find g.

17. $s = \frac{1}{2}gt^2$. Find t.

18. $d = rt$. Find r.

19. $d = rt$. Find t.

20. $W_1L_2 = W_2L_1$. Find L_1.

21. $W_1L_2 = W_2L_1$. Find W_2.

22. $W = Fs$. Find s.

23. $V_1P_1 = V_2P_2$. Find P_2.

24. $V_1P_1 = V_2P_2$. Find P_1.

25. $D(w_1 - w_2) = w_1$. Find w_2.

26. $F = 32 + \frac{9}{5}C$. Find C.

27. $v_2t = v_1t + n$. Find v_1.

28. $v_2t = v_1t + n$. Find t.

29. $v_2t = v_1t + n$. Find n.

30. $s = \frac{1}{2}n(a + l)$. Solve for n; for a; for l.

31. $s = \dfrac{rl - a}{r - 1}$. Solve for r; for l; for a.

32. $a = \frac{1}{2}h(b + b')$. Solve for h; for b; for b'.

33. $C = \dfrac{en}{R + nr}$. Solve for e; for n; for R; for r.

34. There is a formula in physics, relating to moving bodies, that states that $v = at + \frac{1}{2}gt^2$. Solve for a; for g.

35. The volume of a sphere is expressed by the formula $v = \frac{4}{3}\pi r^3$. Find the value of r^3.

36. The volume of a cylinder of radius r and height h is expressed by the formula $v = \pi r^2 h$. Find the value of h; the value of r^2; the value of r.

37. If the side of the square base of a pyramid is s and the height of the pyramid is h, the volume is $v = \frac{1}{3}hs^2$. From this formula find the value of h; the value of s^2; the value of s.

38. The centigrade thermometer is used in scientific work. The number of degrees on the centigrade thermometer can be found from our common (Fahrenheit) thermometer by the formula $C = \frac{5}{9}(F - 32)$, where C is the number of degrees centigrade, and F the number of degrees Fahrenheit. Given $F = 70$, find C. Also find F in terms of C.

Exercise 89. Review Problems

Examples 1 to 4, oral — Examples 5 to 36, written

1. If a square is s feet on a side and a rectangle is n feet longer, how long is the rectangle?

2. If the rectangle in Ex. 1 is m feet narrower than the square, how wide is the rectangle?

3. If $x^2 + 2x - 2 = x^2$, what is the value of x?

4. If $5x^2 + 3x - 21 = 5x^2$, what is the value of x?

5. A rectangle is 3 ft. longer and 2 ft. narrower than a square of the same area. Find the side of the square; the length of the rectangle; the width of the rectangle.

Let x = the number of feet in the side of the square.

Then $x + 3$ = the number of feet in the length of the rectangle,

and $x - 2$ = the number of feet in the width of the rectangle.

$\therefore (x + 3)(x - 2)$ = the area of the rectangle in square feet,

and x^2 = the area of the square in square feet.

$\therefore (x + 3)(x - 2) = x^2$.

Solving, $x = 6$, the number of feet in the side of the square.

$\therefore x + 3 = 9$, the number of feet in the length of the rectangle,

and $x - 2 = 4$, the number of feet in the width of the rectangle.

6. A rectangle is l feet longer and n feet narrower than a square of the same area. Find the side of the square; the length of the rectangle; the width of the rectangle.

7. The length of a floor exceeds the width by 2 ft. If each dimension were 4 ft. more, the area would be 136 sq. ft. more. Find the dimensions.

8. The length of a floor exceeds the width by f feet. If each dimension were n feet more, the area would be s square feet more. Find the dimensions. Evaluate for $f = 2$, $n = 4$, $s = 136$.

9. The length of a rectangular field is 10 rd. more than the width. If the width were 6 rd. more and the length were 5 rd. more, the area would be 200 sq. rd. more. Find the dimensions of the field.

10. Two stations are 300 mi. apart. From the first of these stations a train traveling 30 mi. an hour leaves at the same time that a train traveling 40 mi. an hour leaves the second station. How far apart will the trains be at the end of 3 hr. if they travel toward each other?

Let x = the number of miles they are apart after 3 hr.

Since they approach at the rate of $(30 + 40)$ miles an hour, in 3 hr. they are $3 \cdot (30 + 40)$ miles nearer each other.

$$\therefore 3 \cdot (30 + 40) + x = 300.$$

Solving, $$x = 90.$$

11. In Ex. 10, if the trains travel away from each other, how far apart will they be in 3 hr.?

12. Two stations are d miles apart. From the first of these stations a train traveling r_1 miles an hour leaves at the same time that a train traveling r_2 miles an hour leaves the second station. How far apart will the trains be at the end of t hours if they travel toward each other?

13. In Ex. 12, if the trains travel away from each other, how far apart will they be in t hours?

14. What is the result in Ex. 12 if $r_1 = 30$, $r_2 = 40$, $t = 3$, and $d = 300$? also, if $d = 210$?

15. What is the result in Ex. 12 if $r_1 = 30$, $r_2 = 40$, $t = 3$, and $d = 240$? also, if $d = 180$?

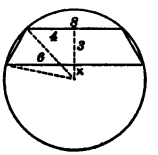
16. What is the result in Ex. 13 if $r_1 = 30$, $r_2 = 40$, $t = 3$, and $d = 300$? also, if $d = 210$?

17. What is the result in Ex. 13 if $r_1 = 30$, $r_2 = 40$, $t = 3$, and $d = 240$? also, if $d = 180$?

18. What are the results in Exs. 12 and 13 if $r_1 = 30$, $r_2 = 30$, $t = 2$, and $d = 60$?

19. A man walking 3 mi. an hour has 5 mi. the start of one walking 4 mi. an hour. How long will it take the second to overtake the first? If the first walks r_1 miles an hour and the second r_2 miles an hour ($r_1 < r_2$), and the first has m miles the start, how long will it take the second to overtake the first?

20. The bases of a trapezoid inscribed in a circle are 12 in. and 8 in., respectively, and the altitude is 3 in. Find the distance from the center to the lower base, and find the radius.



Show that $x^2 + 6^2 = (x + 3)^2 + 4^2$.

21. A caterer prepares a dinner for 50 people at a cost of \$50 to himself. How much must he charge per plate to be sure of making \$17.80 if only 45 people come, counting the unused food as worth 20¢ per plate?

22. A boy walked from A to B at the rate of $3\frac{1}{2}$ mi. per hour. He returned immediately on horseback at the rate of $4\frac{2}{3}$ mi. an hour. Upon his return he found that he had been gone $3\frac{1}{2}$ hr. How far is it from A to B?

23. A man purchases ice at 25¢ per 100 lb. At what rate must he sell it after it has lost 10% of its weight by melting, so that he may gain 17%?

24. A man purchases ice at c cents per 100 lb. At what rate must he sell it after it has lost $r\%$ of its weight by melting, so that he may gain $g\%$?

25. Calculate the side of an equilateral triangle of which the altitude is h. Write a formula for the area of such a triangle.

26. Calculate the sides of an isosceles triangle of which the perimeter is $2p$ and the altitude is h. Evaluate the result for $p = 4$, $h = 2$.

27. A man pays \$60 a month for his city apartment and \$20 a month for service, the heating being furnished free. If he moves to the suburbs, his car fare will be \$8 a month, his coal bill will average \$7 a month, and his service will cost one third as much as his rent. How much rent can he afford to pay in the suburbs so that the total expense per month will be the same as in the city?

28. Alcohol is sold 95% pure. How much water must be added to 1 qt. of such alcohol so that the mixture will be 75% pure?

Let x = the number of quarts to be added.

Then $0.75(1 + x)$ = the number of quarts of alcohol in the mixture.

But 0.95 = the original number of quarts of alcohol, and none has been added.

$\therefore\ 0.75(1 + x) = 0.95.$

Dividing by 0.75,

$$1 + x = 1\tfrac{2}{15}.$$

Subtracting 1, $x = \tfrac{4}{15}$.

Therefore $\tfrac{4}{15}$ qt. must be added.

29. How much water must be added to 1 oz. of a 10% solution of a certain medicine to reduce it to a 2% solution?

30. How many ounces of pure silver must be melted with 75 oz. of silver 750 fine (750 parts of pure silver in 1000 parts of metal) to make a piece of silver 900 fine?

31. How many ounces of gold must be melted with 20 oz. of gold 16 carats fine ($\tfrac{16}{24}$ pure) to make a piece of gold 20 carats fine?

32. How many quarts of milk containing 5% butter fat must be added to 1 qt. of cream containing 35% butter fat so that the mixture will contain 25% butter fat?

33. In a certain alloy of metal weighing 25 lb., 18.75% is pure silver. How many pounds of copper must be melted with it so that the new alloy will contain $15\tfrac{5}{8}$% pure silver?

34. How many pounds of pure water must be added to 25 lb. of sea water containing 16% of salt so that the mixture will contain 2% of salt?

35. What per cent of water must be evaporated from a 6% solution of salt so that the remainder will be a 10% solution?

36. What per cent of water must be evaporated from an r% solution of salt so that the remainder will be an r'% solution?

CHAPTER XIII

SIMULTANEOUS SIMPLE EQUATIONS

133. Indeterminate Equations. If we have one simple equation containing one unknown quantity, we can solve by the methods already studied. If it contains two unknown quantities, we can find one quantity in terms of the other.

For example, if $2x - 3y = 8$, we can see that $x = \frac{8 + 3y}{2}$, and that $y = \frac{2x - 8}{3}$; but this does not tell us the value of either.

Indeed, if $y = 1$ we see that $x = 5\frac{1}{2}$; if $y = 2$, $x = 7$; if $y = 4$, $x = 10$, and so on, there being an indefinite number of values of x and y that satisfy this equation.

An equation that has an indefinite number of roots is called an *indeterminate equation.*

An equation that is not indeterminate is said to be *determinate.*

134. Two Simple Equations. If we have two simple equations containing two unknown quantities, we can usually derive from them a single equation containing only one unknown quantity.

For example, suppose $x + y = 17$,

and $x - y = 9$.

Adding member for member, $2x = 26$;

whence $x = 13$.

And because $x + y = 17$,

we have $13 + y = 17$;

whence $y = 4$.

135. Simultaneous Equations. Two or more equations that have the same values for the unknown quantities are called *simultaneous equations.*

136. Elimination. The process by which we cause an unknown quantity to disappear in deriving an equation from a system of equations is called *elimination.*

Thus in § 134 we derived the equation $2x = 26$ from the system of equations $x + y = 17$ and $x - y = 9$, thereby eliminating y.

The most common methods of elimination are (1) by addition (or subtraction as a special case), and (2) by substitution. These will now be considered.

137. Elimination by Addition or Subtraction. This process is best understood by studying two examples.

1. Solve the system of equations

$$2x + 3y = 27 \quad (1)$$
$$5x - 2y = 1 \quad (2)$$

Multiplying (1) by 2, $4x + 6y = 54$
Multiplying (2) by 3, $15x - 6y = 3$
Adding, $19x = 57.$
Dividing by 19, $x = 3.$
Substituting 3 for x in (1), $6 + 3y = 27.$
Subtracting 6, $3y = 21.$
Dividing by 3, $y = 7.$

Check. Substituting 3 for x, and 7 for y, in (1) and (2), we have

$$6 + 21 = 27,$$
$$15 - 14 = 1.$$

Because y was eliminated by adding two equations, member for member, we say that we have eliminated y by *addition.*

2. Solve the system of equations

$$3x + 2y = 23 \quad (1)$$
$$2x + 3y = 27 \quad (2)$$

Multiplying (1) by 3, $9x + 6y = 69$ (3)
Multiplying (2) by 2, $4x + 6y = 54$ (4)
Subtracting (4) from (3), $5x = 15$
Dividing by 5, $x = 3.$
Substituting in (1) or (2), $y = 7.$

In this solution y is eliminated by *subtraction.*

138. Directions for Elimination. From the preceding solutions we see that in the elimination of an unknown quantity by addition or subtraction we proceed as follows:

Multiply both members of the equations by such numbers as will make the coefficients of one of the unknown quantities numerically equal.

If these coefficients have opposite signs, add the equations member for member; if they have the same signs, subtract.

Solve the resulting equation.

Substitute the result thus found in the simpler of the two given equations and solve for the other unknown quantity.

Check the results by substituting in both of the given equations.

We commonly say "Multiply the equation," meaning thereby "Multiply both members of the equation." Similarly, we speak of adding equations and subtracting equations, meaning that we add or subtract member for member.

Exercise 90. Elimination by Addition or Subtraction

Examples 1 to 8, oral — Examples 9 to 38, written

1. Solve for x,
$$x + y = 9$$
$$x - y = 3$$

2. Solve for y,
$$x + 2y = 7$$
$$x + y = 4$$

3. Solve for x,
$$3x + 2y = 5$$
$$5x - 2y = 3$$

4. Solve for m,
$$7m + 3n = 23$$
$$5m + 3n = 19$$

5. Solve for u,
$$2u + v = 14$$
$$3u - v = 11$$

6. Solve for w,
$$2w + 3z = 20$$
$$w - 3z = 1$$

7. Solve for P,
$$3P + Q = 17$$
$$2P - Q = 8$$

8. Solve for F,
$$2F + 7G = 21$$
$$3F - 7G = 14$$

Solve the following by addition or subtraction:

9. $2x + 3y = 13$
$5x - 4y = 16$

10. $2x + 5y = 51$
$5x + 2y = 54$

11. $6x + 5y = 33$
$7x - 2y = 15$

12. $9x + 7y = 16$
$3x - 2y = 1$

13. $9x + 2y = 24$
$7x + 3y = 23$

14. $4x + y = 23$
$5x - y = 13$

15. $7x + 6y = 20$
$5x - 2y = 8$

16. $4x + 9y = 3$
$3x + 7y = 2$

17. $3x + 8y = 38$
$7x - 2y = 6$

18. $5x - 3y = 20$
$3x - 4y = 1$

19. $4x + 3y = 18$
$3x - 2y = 5$

20. $5x + 2y = 32$
$3x + y = 18$

21. $2x + 7y = 34$
$7x + 2y = 74$

22. $4x - 7y = 20$
$7x - 4y = 68$

23. $21x + 11y = 288$
$30x - 16y = 126$

24. $0.2x + 0.3y = 12$
$0.5x + 0.4y = 23$

25. $0.7x + 0.5y = 53$
$0.9x - 0.4y = 16$

26. $1.5x + 1.25y = 19$
$1.2x + 0.75y = 13.2$

27. $u + 4v = 31$
$4u + v = 19$

28. $8v + 5w = 59$
$7v + 2w = 35$

29. $5m - 2n = 23$
$13m - 3n = 51$

30. $6p + 7q = 31$
$3p + 2q = 5$

31. $11s - 12t = 31$
$10s + 7t = 64$

32. $4b - 3h = 29$
$2b - h = 47$

33. $10r_1 + 3r_2 = 75$
$8r_1 - r_2 = 43$

34. $6\frac{2}{7}r + 3\frac{1}{7}r' = 110$
$9\frac{3}{7}r + 3\frac{1}{7}r' = 154$

35. $1.2x - y = 0.2$
$3x - 3\frac{1}{3}y = 4$

36. $3r + 4r' = 63$
$9r - 3r' = 54$

37. $8D + 3E = 77$
$7D - 2E = 35$

38. $T + M = 100$
$2T - 0.1M = 139.1$

139. Elimination by Substitution. It is often convenient to find from one equation the value of one unknown quantity in terms of the other, and to substitute this in the other equation.

1. Solve the system of equations

$$3x + 7y = 22.4 \quad (1)$$

$$x - 5y = 6 \quad (2)$$

From (2), $x = 6 + 5y.$ (3)

Substituting in (1), $3(6 + 5y) + 7y = 22.4,$

or $18 + 15y + 7y = 22.4.$

Subtracting 18, $22y = 4.4.$

Dividing by 22, $y = 0.2.$

Substituting in (3), $x = 6 + 5 \times 0.2$

$= 6 + 1 = 7.$

Check. $3 \times 7 + 7 \times 0.2 = 22.4,$

and $7 - 5 \times 0.2 = 6.$

This plan is sometimes advantageous when the coefficient of one of the unknown quantities is 1.

2. Solve the system of equations

$$5x + 2y = 34 \quad (1)$$

$$7x - 3y = 7 \quad (2)$$

From (1), $y = \frac{34 - 5x}{2}.$ (3)

Substituting in (2), $7x - 3 \cdot \frac{34 - 5x}{2} = 7.$

Multiplying by 2, $14x - 102 + 15x = 14.$

Solving, $x = 4.$

Substituting in (3), $y = \frac{34 - 5 \cdot 4}{2}$

$= 17 - 5 \cdot 2 = 7.$

3. Solve the system of equations

$$x + 3y = 7$$

$$x - 2y = 2$$

We have $x = 7 - 3y$ and $x = 2 + 2y$; hence $7 - 3y = 2 + 2y$, and $y = 1.$ $\therefore x = 4.$ This special form of elimination by substitution is sometimes called *elimination by comparison.*

140. Directions for Elimination. Therefore, to eliminate by substitution,

From one of the equations find the value of either unknown quantity in terms of the other.

Substitute this value in the other equation and solve.

Exercise 91. Elimination by Substitution

Examples 1 to 7, oral — Examples 8 to 21, written

1. If $x = 3$ and $x + y = 5$, what is the value of y?
2. If $y = 7$ and $x + y = 16$, what is the value of x?
3. If $x = -2$ and $x + y = 1$, what is the value of y?
4. If $y = -2$ and $x + y = 0$, what is the value of x?
5. If $r = 4$ and $r + r' = 9$, what is the value of r'?
6. If $r = 7$ and $r - r' = 5$, what is the value of r'?
7. If $P = 5$ and $P + Q = 9.8$, what is the value of Q?

Solve the following equations by substitution:

8. $x + 7y = 26$
   $2x + 3y = 19$
9. $x - 3y = -1$
   $3x + 2y = 19$
10. $x + 4y = 35$
    $3x - 2y = 7$
11. $x - 4y = 1$
    $5x + 2y = 49$
12. $x + 5y = 25$
    $7x + 3y = 47$
13. $x - 2y = 2$
    $6x + 5y = 80$
14. $2x + 17y = 61$
    $8x - y = 37$
15. $4.5x - 7y = 7$
    $5.5x + 6y = 59.5$
16. $0.8x + 0.3y = 11.3$
    $2x + 3.5y = 14.5$
17. $5m + 7n = 125$
    $7m - n = 13$
18. $5p - 3q = 27$
    $7q - 3p = 15$
19. $19x - y = 18$
    $27x + 4y = 31$
20. $F + 7M = 26$
    $F - 19M = -52$
21. $2E + 6F = 15.4$
    $E + 8.8F = 8.86$

Exercise 92. Problems

Examples 1 to 9, oral — Examples 10 to 28, written

1. Five more than three times a number is 20. What is the number?

2. The sum of two numbers is 34. One of the numbers is 19. What is the other number?

3. The difference of two numbers is 17. The larger number is 32. What is the smaller number?

4. The difference of two numbers is 41. The smaller number is 19. What is the larger number?

5. The difference of two numbers is 12. One of the numbers is 20. What is the other number? (Note that there are two possible answers.)

6. The product of two numbers is 51. One of the numbers is 17. What is the other number?

7. The quotient of two numbers is 27. The divisor is 3. What is the dividend?

8. The quotient of two numbers is 32. The dividend is 96. What is the divisor?

9. The quotient of two numbers is 10. One of the numbers is 20. What is the other number? (Note that there are two possible answers.)

10. The sum of two numbers is 32, and one of the numbers is three times the other. Find the numbers.

11. The sum of two numbers is 36, and one of the numbers is two more than the other. Find the numbers.

12. The sum of two numbers is 36, and one of the numbers is two less than the other. Find the numbers.

13. The sum of two numbers is 5.6, and one of the numbers is 4.2 more than the other. Find the numbers.

14. The sum of two numbers is 26.1, and one of the numbers is 11.96 more than the other. Find the numbers.

15. If a rectangle were 2 in. longer and 3 in. wider, its area would be increased by 35 sq. in. If it were 2 in. shorter and 2 in. wider, the area would be unchanged. Find the dimensions.

Let l = the number of inches of length,
and w = the number of inches of width.
Then lw = the number of square inches of area.
Then $(l + 2)(w + 3) = lw + 35$,
and $(l - 2)(w + 2) = lw$.
Simplifying, $3l + 2w = 29$,
and $l - w = 2$.
Solving, $l = 6.6$,
and $w = 4.6$.

Therefore the dimensions are 6.6 in. and 4.6 in.

Here the initial letters of length and width have been used to represent the unknown quantities, a custom that is coming into use. It is permissible to use x and y, l and w, or any convenient letters.

16. If a rectangle were 5 in. shorter and 1 in. wider, its area would be decreased by 35 sq. in. If it were 5 in. longer and 1 in. wider, its area would be increased by 65 sq. in. Find the dimensions.

17. If the perimeter of a rug is 20 ft., and if three times the length plus five times the width is 36 ft., what is the area?

18. The length of a room is $33\frac{1}{3}\%$ greater than the width, and the perimeter is 70 ft. Find the dimensions.

19. The width of a room is two thirds the length, and the length exceeds the width by 7 ft. Find the dimensions.

20. The length of a rectangular swimming tank is 50% greater than the width, and the perimeter is 160 ft. Find the dimensions.

21. The altitude of a rectangle is 20% less than the base, and the perimeter is 18 in. Find the dimensions.

22. The circumference of a circle is 10.708 in. greater than the diameter. Find the circumference and the diameter. (From § 15, $c = 3.1416\,d$.)

23. A bird flying with the wind travels 50 mi. an hour, but when flying against a wind that is twice as strong, it travels only 20 mi. an hour. Find the rate of the wind for each case. Find the rate of the bird in still air.

In such problems it is assumed that the rate of the wind should be added to the rate of the moving body when the latter goes with the wind, and subtracted when it goes against the wind. This is approximately the case in the problems considered.

Taking the rates to mean the number of miles per hour,

let $x =$ the rate of the bird,

and $y =$ the rate of the wind the first time.

Then $x + y = 50,$

and $x - 2y = 20.$

Solving, $x = 40,$

and $y = 10.$

$\therefore 2y = 20,$ the rate of the wind the second time.

24. An aëroplane flies with the wind at the rate of 75 mi. an hour, and against the wind at the rate of 45 mi. an hour. Find the rate of the wind. Find the rate of the aëroplane in still air.

25. A river steamer can go 26 mi. an hour with the current and 14 mi. an hour against it. Find the rate of the current. Find the rate of the steamer in still water.

26. The report of a gun was heard in 3 sec. at a place 3189 ft. distant, toward which the wind was blowing, and in 2 sec. at a place 2074 ft. distant, from which the wind was blowing. Find the velocity of sound and the rate at which the wind was blowing.

27. The report of a gun traveled 357⅓ yd. a second with the wind and 346 yd. a second against the wind. Find the velocity of sound and the rate at which the wind was blowing.

28. A conductor walking from the rear to the front of a moving express train passes a telegraph pole at the rate of 45.2 mi. an hour. Walking to the rear of the train, he passes a telegraph pole at the rate of 40.6 mi. an hour. How fast is the train going, and how fast is the conductor walking?

141. Simultaneous Equations containing Fractions. If the equations contain fractions, it is usually better to clear of fractions before attempting to eliminate.

Solve the equations

$$\frac{x+2}{x-2}=\frac{y+8}{y+4} \tag{1}$$

$$\frac{2x-5}{4y-1}=\frac{x-1}{2y+1} \tag{2}$$

Clearing (1), $xy+2y+4x+8=xy-2y+8x-16.$ (3)

Clearing (2), $4xy-10y+2x-5=4xy-4y-x+1.$ (4)

Simplifying (3), $x-y=6.$

Simplifying (4), $x-2y=2.$

Solving, $x=10,$

and $y=4.$

Check. $\frac{10+2}{10-2}=\frac{4+8}{4+4}=\frac{12}{8};$

$\frac{20-5}{16-1}=\frac{9}{9}.$

Exercise 93. Simultaneous Equations containing Fractions

Examples 1 to 5, oral — Examples 6 to 21, written

1. If $\frac{1}{x}=3$, what is the value of x?

2. If $\frac{3}{x}=7$, what is the value of x?

3. Find the value of x when $\frac{1}{x}=\frac{1}{7}$; when $\frac{2}{x}=\frac{3}{7}$. Check the results.

4. If $\frac{1}{x}+\frac{1}{y}=4$ and $\frac{1}{x}-\frac{1}{y}=2$, what is the value of $\frac{2}{x}$? of $\frac{1}{x}$? of x? Check the result.

5. If $\frac{1}{x}+\frac{2}{y}=4$ and $\frac{1}{x}-\frac{2}{y}=0$, what is the value of $\frac{2}{x}$? of $\frac{1}{x}$? of x? of y? Check the results.

Solve the following equations:

6. $\frac{x}{3}+\frac{y}{4}=4$
$\frac{x}{4}+\frac{y}{3}=\frac{25}{6}$

7. $\frac{x+y}{3}=\frac{x-y}{5}$
$\frac{x-3y}{7}=2x+7$

8. $\frac{x-4}{x-3}=\frac{y+4}{y+7}$
$\frac{x+5}{x+2}=\frac{y-1}{y-2}$

9. $\frac{x+y}{8}+\frac{x-y}{6}=5$
$\frac{x+y}{4}-\frac{x-y}{3}=10$

10. $\frac{5}{x+2y}=\frac{7}{2x+y}$
$\frac{7}{3x-2}=\frac{5}{6-y}$

11. $\frac{3x-2}{5x-1}=\frac{3y+7}{5y+16}$
$\frac{3x-1}{x+5}=\frac{6y-5}{2y+3}$

12. $\frac{4x+5y}{40}=x-y$
$\frac{2x-y}{3}=\frac{1-4y}{2}$

13. $\frac{x+y}{3}+x=15$
$\frac{x-y}{5}+y=6$

14. $\frac{x+y}{7}+\frac{x-y}{2}=1$
$\frac{x+y}{2}-\frac{x-y}{8}=7\frac{1}{4}$

15. $\frac{x+2y+1}{2x-y+1}=2$
$\frac{3x-y+1}{x-y+3}=5$

16. $\frac{x+y}{3}+\frac{y-x}{2}=9$
$\frac{x}{2}+\frac{x+y}{9}=5$

17. $7x-\frac{y}{5}=48$
$\frac{35y+x}{7}=26$

18. $\frac{x+y-1}{x-y+1}=7$
$\frac{y-x+1}{x-y+1}=35$

19. $\frac{x+y+3}{x-y-3}=-\frac{3}{2}$
$\frac{x-y-3}{x-y+3}=-2$

20. $\frac{x-y}{x+y}=-\frac{8}{15}$
$\frac{63x-3y-44}{7}=100$

21. $\frac{x+3y+13}{4x+5y-25}=3$
$\frac{8x+y+6}{5x+3y-23}=5$

Exercise 94. Problems

Examples 1 to 6, oral — Examples 7 to 31, written

1. Find the value of x, given: $\frac{1}{x}=7$; $\frac{1}{x}=1\frac{1}{2}$; $\frac{1}{x}=\frac{1}{5}$; $\frac{1}{x}=0.2$.

2. Find the value of k, given: $\frac{1}{k}=\frac{1}{3}$; $\frac{1}{k}=\frac{2}{3}$; $\frac{1}{k}=\frac{3}{4}$; $\frac{1}{k}=0.25$.

3. Find the value of r, given: $2\pi r=\pi$; $2\pi r=4\pi$; $2\pi r=\frac{1}{\pi}$.

4. Given $\frac{1}{x}+\frac{1}{y}=6$ and $\frac{1}{x}-\frac{1}{y}=4$, find the value of x.

5. The reciprocal of what number equals 9?

6. If one more than the reciprocal of a certain number equals $\frac{4}{3}$, what is the number?

7. If to the larger of two numbers we add 8, and then divide by the smaller number, the quotient is 2; but if from the larger we subtract 2, and then divide by the smaller, the quotient is 1. Find the numbers.

8. The larger of two numbers lacks one of being six times the smaller number; and if 2 is subtracted from the larger, and the remainder is divided by the smaller, the quotient is 5. Find the numbers.

9. A certain fraction $\frac{x}{y}$ becomes equal to $\frac{1}{2}$ if 3 is added to its numerator and 1 to its denominator, and equal to $\frac{1}{3}$ if 1 is added to its numerator and 1 is subtracted from its denominator. Find the fraction.

10. A certain fraction becomes equal to $\frac{2}{3}$ if 1 is added to its numerator and 2 is subtracted from its denominator, and equal to $\frac{1}{2}$ if 1 is subtracted from its numerator and 4 from its denominator. Find the fraction.

11. A certain fraction becomes equal to $\frac{1}{3}$ if 3 is subtracted from its numerator and from its denominator, and equal to $\frac{3}{5}$ if 1 is added to its numerator and to its denominator. Find the fraction.

12. A boatman rows 5 mi. down a river and back in 2 hr. He can row 1 mi. down the river in the same time that he can row 0.6 mi. up the river. Find the time he rows down and up respectively.

Let $x =$ the number of hours going down,

and $y =$ the number of hours going up.

Then $x + y = 2.$

Also, since the time to row a mile downstream is 0.6 of the time to row a mile upstream, and so for any other distance,

$$x = 0.6y.$$

Solving, $x = \frac{3}{4}$, and $y = 1\frac{1}{4}$.

13. A boat's crew can row down a river at the rate of 10 mi. an hour, and up the river at the rate of 5 mi. an hour. Find the rate of the stream, and their rate in still water.

14. A steam launch goes 20 mi. upstream and 36 mi. downstream in 8 hr. It can go 3 mi. downstream in the same time that it can go 1 mi. upstream. Find the rate of the stream, and the rate of the boat in still water.

Let $b =$ the rate of the boat, in miles per hour,

and $s =$ the rate of the stream.

Then $$\frac{20}{b-s} + \frac{36}{b+s} = 8,$$

and $$\frac{3}{b+s} = \frac{1}{b-s}.$$

Explain how these two equations are derived.

Solve the equations and check the results.

15. A steamer can go downstream at the rate of 22 mi. an hour, but its rate upstream is only $\frac{7}{11}$ as fast. Find the rate of the stream, and the rate of the steamer in still water.

16. Two bodies are 96 yd. apart. If they move toward each other with uniform (but unequal) rates, they will meet in 8 sec.; but if they move in the same direction, the swifter overtakes the slower in 48 sec. Find the rate of each.

17. When weighed in water, tin loses 0.137 of its weight, and copper 0.112 of its weight. If a 10-pound mass of tin and copper loses 1.195 lb., find the weight of the tin and the copper in the mass.

Let t = the number of pounds of tin,

and c = the number of pounds of copper.

Then $t + c = 10,$ (1)

and $0.137t + 0.112c = 1.195.$ (2)

Multiplying (1) by 0.112,

$0.112t + 0.112c = 1.12.$

Subtracting, $0.025t = 0.075.$

Dividing by 0.025, $t = 3.$

Whence $c = 7.$

Therefore there are 3 lb. of tin and 7 lb. of copper in the mass.

18. When weighed in water, silver loses 0.095 of its weight, and copper 0.112 of its weight. If a 12-pound mass of silver and copper loses 1.174 lb., find the weight of the silver and the copper in the mass.

19. When weighed in water, gold loses 0.051 of its weight, and silver 0.095 of its weight. If a 6-ounce piece of gold and silver loses 0.35 oz., find the weight of the gold and the silver in the piece.

20. When weighed in water, tin loses 0.137 of its weight, and lead 0.089 of its weight. If a 65-pound mass of tin and lead loses 6.025 lb., find the weight of the tin and the lead in the mass.

21. An iron bar covered with brass weighs 13 lb. When weighed in water, iron loses 0.128 of its weight, and brass 0.119 of its weight. The bar loses 1.655 lb. when weighed in water. Find the weight of the brass that covers the iron.

22. In water, 1 oz. of platinum weighs only 0.9535 oz., and 1 oz. of gold 0.949 oz. An ingot of gold and platinum that weighs 6 oz. in air weighs 5.6985 oz. in water. Find the weight of the gold and the platinum.

23. Twenty-seven coins, dollars and quarters, amount to \$19.50. How many are there of each kind?

Let d = the number of dollars,

and q = the number of quarters.

Then $d + q = 27$,

and $d + \frac{q}{4} = 19.50$.

Explain the second equation and solve.

24. A man distributed \$5.25 in the form of dimes and quarters among thirty boys, each boy receiving one coin. How many boys received dimes? How many received quarters?

25. A man distributed \$3.50 in the form of dimes and nickels among fifty children, each child receiving one coin. How many received dimes? How many received nickels?

26. The admission to an entertainment was 50¢ for adults and 25¢ for children. The proceeds from 125 tickets were \$51.25. How many adults were admitted? How many children?

27. A school gave an entertainment at which the tickets to pupils were sold at 40¢ each, and to others at 50¢ each. There were 245 tickets sold and the receipts were \$108.50. How many of each kind were sold?

28. A grocer has in his cash drawer 106 bills, some one-dollar and the rest two-dollar bills. The total amount is \$138. How many has he of each?

29. A receiving teller at a bank took in 515 bills, some five-dollar and the rest two-dollar bills. The total amount was \$2155. How many of each did he receive?

30. A paymaster at a shop has 210 silver pieces, some quarters and the rest half dollars. The total amount is \$75. How many has he of each?

31. A dealer has 19 pieces of iron pipe, some 12 ft. long and the rest 6 ft. The total length of the pipe is 168 ft. How many pieces has he of each?

142. Literal Simultaneous Equations. Although it is coming to be a custom to use initial letters for the unknown quantities in numerical equations, it is still common to employ the first letters of the alphabet for numbers supposed to be known, and the last letters for those supposed not to be known, in literal equations.

Solve the equations

$$ax + by = c \tag{1}$$

$$a'x + b'y = c' \tag{2}$$

Here a and a' are entirely different, it being convenient to use similar forms as coefficients of the same letter x.

Multiplying (1) by b', $\quad ab'x + bb'y = b'c$

Multiplying (2) by b, $\quad a'bx + bb'y = bc'$

Subtracting, $\quad (ab' - a'b)x = b'c - bc'$

Dividing by $ab' - a'b$, $\quad x = \dfrac{b'c - bc'}{ab' - a'b}$.

It now remains to find the value of y. This may be done as in the case of x, or we may substitute the value of x in one of the given equations. Substituting in (1), we have

$$a \cdot \frac{b'c - bc'}{ab' - a'b} + by = c.$$

Subtracting $a \cdot \dfrac{b'c - bc'}{ab' - a'b}$, $\quad by = c - a \cdot \dfrac{b'c - bc'}{ab' - a'b}$

$$= \frac{ab'c - a'bc - ab'c + abc'}{ab' - a'b}$$

$$= \frac{b(ac' - a'c)}{ab' - a'b}.$$

Dividing by b, $\quad y = \dfrac{ac' - a'c}{ab' - a'b}$.

We might have found y by noticing that if we interchanged x and y in the equations, we had to interchange a and b and also a' and b'. We may therefore write the result for y by making these changes in the value of x.

The teacher should use his discretion as to the amount of checking to be required in the case of literal equations. While the only complete check is that of substitution in the original equations, this is often so tedious that the teacher may prefer to tell whether the result is correct.

Exercise 95. Literal Simultaneous Equations

Examples 1 to 6, oral — Examples 7 to 21, written

1. Solve the equations: $x + y = a$, $x - y = b$.
2. Solve the equations: $x + y = 3a$, $x - y = a$.
3. Solve the equations: $2x + y = a$, $2x - y = b$.
4. Solve the equations: $2x + y = 3a$, $2x - y = a$.
5. Solve the equations: $x + y = 4a$, $y = a$.
6. Solve the equations: $x + y = 2a + b$, $x - y = 2a - b$.

Solve the following equations:

7. $x + y = 15a + b$
   $x - y = 15a - b$

8. $ax + y = m$
   $bx - y = n$

9. $ax + by = k$
   $cx - dy = l$

10. $a^2x + b^2y = c^2$
    $p^2x + q^2y = r^2$

11. $x + y = \dfrac{2(a^2 + b^2)}{a^2 - b^2}$
    $x - y = \dfrac{4ab}{a^2 - b^2}$

12. $\dfrac{x + y + 1}{x + y - 1} = m$
    $\dfrac{x - y + 1}{x - y - 1} = n$

13. $\dfrac{x - a + c}{y - a + b} = \dfrac{b}{c}$
    $\dfrac{x + c - a}{y + a - b} = \dfrac{c}{b}$

14. $ax + by = 3ab$
    $a^2x + b^2y = a + b$

15. $ax + by = c$
    $bx + ay = d$

16. $ax - by = cd$
    $bx - ay = ef$

17. $abx + cdy = k$
    $pqx + mny = l$

18. $\dfrac{x + y}{x - y} = \dfrac{a}{b - c}$
    $\dfrac{x + c}{y + b} = \dfrac{a + b}{a + c}$

19. $\dfrac{x + 1}{x - 1} + \dfrac{y - 1}{y + 1} = 3$
    $\dfrac{x + 1}{x - 1} - \dfrac{y - 1}{y + 1} = 3$

20. $\dfrac{x + a}{y + b} = \dfrac{a + b + c}{a - b + c}$
    $\dfrac{x - a}{y - b} = \dfrac{a + b - c}{a - b + c}$

21. $(a + b + c + d)x + (a - b + c + d)y = a + b - c + d$
    $(a - b - c + d)x - (a - b + c + d)y = a - b + c + d$

143. Three Simultaneous Equations. If three or more simultaneous equations are given, involving three or more unknown quantities, the quantities are eliminated by combining pairs of equations, as shown in the following solution.

Solve the equations

$$5x + 2y - 4z = -3 \quad (1)$$

$$3x - 3y + 5z = 12 \quad (2)$$

$$4x + 5y + 2z = 20 \quad (3)$$

We may eliminate z by combining (1) and (2), as follows:

Multiplying (1) by 5, $25x + 10y - 20z = -15.$ (4)

Multiplying (2) by 4, $12x - 12y + 20z = 48.$ (5)

Adding, $37x - 2y = 33.$ (6)

We may eliminate z between (1) and (3), as follows:

Multiplying (3) by 2, $8x + 10y + 4z = 40.$ (7)

Adding (1) and (7), $13x + 12y = 37.$ (8)

We now have two equations, (6) and (8), involving x and y. We may now eliminate y as follows:

Multiplying (6) by 6, $222x - 12y = 198.$ (9)

Adding (8) and (9), $235x = 235.$

$\therefore x = 1.$

Substituting in (8), $13 + 12y = 37.$

Hence $12y = 24,$

and $y = 2.$

Substituting $x = 1$, $y = 2$, in (1), $5 + 4 - 4z = -3.$

Solving for z, $z = 3.$

Therefore $x = 1, y = 2, z = 3.$

Check. Substituting in (1), (2), and (3), we have

$$5 + 4 - 12 = -3,$$

$$3 - 6 + 15 = 12,$$

$$4 + 10 + 6 = 20.$$

In checking, it is usually sufficient to substitute in the equation which has been used least in the solution. In this case the equation is (3). To be sure of the results, however, it is necessary to substitute in all three of the original equations.

Exercise 96. Three Simultaneous Equations

Examples 1 to 4, oral — Examples 5 to 24, written

1. In the system of equations

$$x + y + z = 6 \quad (1)$$
$$x + y - z = 4 \quad (2)$$
$$x - y + z = 2 \quad (3)$$

eliminate y and z at the same time from (2) and (3). What is the value of x?

2. Eliminate z from (1) and (2) and find the value of $x + y$. Then substitute the value of x found in Ex. 1, and find the value of y.

3. Substitute the values of x and y found in Exs. 1 and 2, and find the value of z from (1).

4. Check the values of x, y, and z found in Exs. 1–3, by substituting in (2).

Solve the following equations:

5. $x + y + z = 10$
$x - y + z = 2$
$x + y - z = 8$

6. $x + y + z = 7$
$3x + y - z = 3$
$2x + 4y + z = 12$

7. $x + y + z = 13$
$3x + y - 3z = 5$
$x - 2y + 4z = 10$

8. $x + 2y + 3z = 41$
$x - 3y + 4z = 6$
$5x + 6y - 7z = 63$

9. $2x - y + z = 3$
$x + 2y + z = 12$
$4x - 3y + z = 1$

10. $3x + 2y - 4z = 15$
$5x - 3y + 2z = 60$
$2x + 4y - 3z = 45$

11. $x - 3y + z = 10$
$2x - 7y - 5z = -2$
$x + y - 2z = 5$

12. $10x + 8y - 9z = 10$
$12x + 2y - 16z = 15$
$2x + 10y - 25z = 0$

13. $x + y + z = 14.6$
$x - y + z = 3.4$
$x + y - z = 12.2$

14. $x + 2y - z = 3.25$
$3x - y + z = 3$
$x + y - 5z = 1.92$

15. A and B can do a piece of work in 3 da., B and C in 4 da., C and A in 5 da. How long will it take each to do the work?

Let a, b, c = the number of days required by A, B, C, respectively.

Then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ = the parts they can do in 1 da., respectively.

Then $\frac{1}{a}+\frac{1}{b}$ = the part A and B together can do in 1 da.

But $\frac{1}{3}$ = the part A and B together can do in 1 da.

Therefore $$\frac{1}{a}+\frac{1}{b}=\frac{1}{3}. \qquad (1)$$

Similarly, $$\frac{1}{b}+\frac{1}{c}=\frac{1}{4}, \qquad (2)$$

and $$\frac{1}{c}+\frac{1}{a}=\frac{1}{5}. \qquad (3)$$

Adding and dividing by 2,

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{47}{120}.$$

Subtracting (1), $$\frac{1}{c}=\frac{7}{120}.$$

Similarly, $$\frac{1}{b}=\frac{23}{120},$$

and $$\frac{1}{a}=\frac{17}{120}.$$

Solving, $$a=7\tfrac{1}{17}, b=5\tfrac{5}{23}, c=17\tfrac{1}{7}.$$

Therefore it will take A $7\frac{1}{17}$ da., B $5\frac{5}{23}$ da., and C $17\frac{1}{7}$ da.

16. A and B can do a piece of work in 7 da., B and C in 6 da., C and A in 5 da. How long will it take each to do the work?

17. A cistern can be filled by two pipes, X and Y, in 35 min., by X and Z in 42 min., and by Y and Z in 70 min. How long will it take X, Y, and Z to fill it? How long will it take each?

18. A and B can build a wall in p days, B and C in q days, C and A in r days. How long will it take them working together to build the wall?

19. A and B can remove a pile of bricks in $2\frac{1}{2}$ da., B and C in 4 da., C and A in $3\frac{1}{3}$ da. How long will it take them working together to remove the pile? How long will it take each working alone?

20. Three villages, A, B, and C, are situated at the vertices of a triangle. The distance from A to B by way of C is 76 mi.; from A to C by way of B, 79 mi.; from B to C by way of A, 81 mi. Find the direct distance from A to B; from B to C; from C to A.

21. Three boys were playing marbles, when A remarked that if B gave him one of his marbles, B would have twice what A then had. C remarked that if B gave him three marbles, C would have twice what B then had. A then remarked that if C gave him seven of his marbles, the number that A would have would lack three of being half as many as C would have left. How many marbles did each have?

22. A printing office furnishes 1000 cards, 2000 billheads, and 3000 letterheads for \$10.50; 2000 cards, 1000 billheads, and 2000 letterheads for \$8.50; 1000 cards, 1000 billheads, and 2000 letterheads for \$7. Find the cost of each per thousand.

23. In an athletic meet the following was the final score of teams A, B, and C:

	1st Places	*2d Places*	*3d Places*	*Total No. of Points*
A	5	3	2	36
B	2	4	1	23
C	2	2	6	22

How many points did each place count?

The first equation is $5x + 3y + 2z = 36$.

24. A cheese factory received 69,000 lb. of milk in June and July, 53,000 lb. in July and August, and 94,000 lb. in June, July, and August. How many pounds did it receive in each of the months?

Exercise 97. Miscellaneous Problems

Examples 1 to 6, oral — Examples 7 to 24, written

1. In a certain number the sum of the units' and tens' digits is 6, and the tens' digit is half the units' digit. State the equations.

2. The sum of two numbers is 10 and their difference is 6. State the equations.

3. Twice the first of two numbers added to the second equals 9. Twice the second added to the first equals 6. State the equations.

4. If a man can do a piece of work in d days, what part of the work can he do in one day?

5. How will you represent a general number of two digits? of three digits?

6. If A can do a piece of work in a days, and B can do it in b days, what part of the work can they do in one day, working together?

7. A pupil in the arithmetic class was told to add 3 to a certain number and divide the sum by 2. Misunderstanding the problem he subtracted 2 from the number and multiplied by 3, and yet he obtained the correct result. What was the number?

8. A pupil was told to add a to a certain number and divide the sum by b. Misunderstanding the problem he subtracted b from the number and multiplied by a, and yet he obtained the correct result. What was the number? Evaluate the result for $a = 3$, $b = 2$; also for $a = 3$, $b = 1$.

9. Two trains go from A to B over different roads, one of which is 15 mi. longer than the other. The train on the shorter route takes 6 hr., and the one on the longer route, traveling 10 mi. less per hour, takes 8 hr. 30 min. Find the length of each route.

10. A boy riding a bicycle at the rate of 9 mi. an hour is sent to overtake a boy who is riding horseback at the rate of 6 mi. an hour and who had 4 mi. the start. How long will it take the second boy to overtake the first?

11. The cost of publication of each copy of a certain illustrated magazine is $6\frac{1}{2}¢$. It sells to dealers for 6¢, and the amount received for advertising is 10% of the amount received for all the magazines in excess of 10,000. Find the least number of magazines that can be issued without loss.

12. A man bought 10 cows and 50 sheep for $750. He sold the cows at a profit of 10%, and the sheep at a profit of 30%, receiving in all $875. Find the average cost of the cows and the average cost of the sheep.

13. A number of boys purchase a camp. If there had been two more in the company, each would have paid $12 less; and if there had been three less, each would have paid $24 more. How many boys were there and how much did each of them pay?

14. A boat's crew can row 15 mi. an hour downstream. The crew can row a certain distance in still water in 15 min., and requires 20 min. to row the same distance upstream. Find the rate of the stream and the rate of rowing in still water.

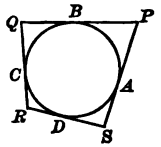
15. The perimeter of a rectangle is 60 ft. If the length is increased by 3 ft. and the width is decreased by 3 ft., the area is decreased by 21 sq. ft. Find the dimensions.

16. The length of a rectangle is twice its width. If the length and width are both increased by 1 in., the area is increased by 31 sq. in. Find the dimensions.

17. If the length of a rectangular rug is increased by 5 ft. and the width is decreased by 2 ft., the area is increased by 10 sq. ft. If the length is increased by 2 ft. and the width is decreased by 5 ft., the area is decreased by 65 sq. ft. Find the dimensions of the rug.

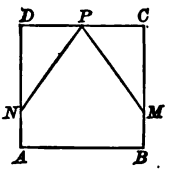
18. In any quadrilateral the sum of the four angles equals 360°. There is a quadrilateral with angles A, B, C, and D, such that $A + B + C = 280°$, $B + C + D = 280°$, and $C + D = 190°$. Find the size of each angle.

19. In this figure the following equalities are known: $PA = PB$, $QB = QC$, $RC = RD$, $SA = SD$. It is further known that $PS = 9$, $PQ = 8$, $QR = 5$, and $PA = SA + 1$. Find the lengths of SA, AP, PB, BQ, QC, CR, RD, DS.

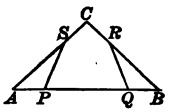


20. A box contains a mixture of 6 qt. of oats and 9 qt. of corn, and another box contains a mixture of 6 qt. of oats and 2 qt. of corn. How many quarts must be taken from each box to have a mixture of 7 qt., half of which will be oats and half corn?

21. In this square M, P, and N are so taken that $BM = \frac{1}{2}MC$, and $CP = PD$. The perimeter of the square is 40. Find the lengths of BM, MC, and CP.



22. In this triangle $AC = BC$ and $AP = QB = RC = CS$. It is also given that $AQ = 3\frac{1}{2}$, $BC = 3$, and the perimeter of the triangle is $10\frac{1}{2}$. Find the lengths of AP, PQ, and BR.



23. A person has \$18,375 to invest. He can buy 3% bonds at 75 (a \$100 bond that pays 3% interest being purchased by him for \$75), and 5% bonds at 120. How much of his money must he invest in each kind of bond to have the same income from each investment?

24. The formula for the area of a trapezoid is $a = \frac{1}{2}h(b + b')$, where h is the height and b and b' are the bases. It is known that a certain trapezoid with a height of 10 in. has an area of 140 sq. in., and that the upper base of the trapezoid is 4 in. shorter than the lower base. Find the length of each base of the trapezoid.

CHAPTER XIV

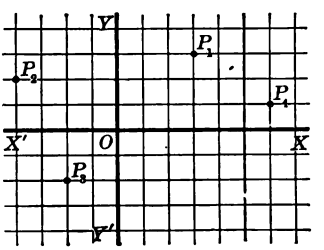
GRAPHS

144. Location of Points on a Map. Points are located on a map by means of latitude and longitude. Latitude is stated in degrees north or south of the equator, longitude in degrees east or west of the prime meridian through Greenwich, England.

Thus to the nearest degree, the position of New York is 41° N. (that is, the latitude is 41° north of the equator) and 74° W. (that is, the longitude is 74° west of Greenwich). Similarly, the position of Chicago is 42° N. and 88° W.; of San Francisco, 38° N. and 122° W.; and of Paris, 49° N. and 2° E.

145. Location of Points on Paper. In a similar way we may locate points on paper. We may take two lines, one vertical and the other horizontal, and measure distances to the right and left of the one, and up and down from the other.

In this figure, P_1 is 3 units to the right of the vertical line YY' and 3 units above the horizontal line XX'; P_2 is 4 units to the left of YY' and 2 units above XX'; P_3 is 2 units to the left and 2 units below; and P_4 is 6 units to the right and 1 unit above.

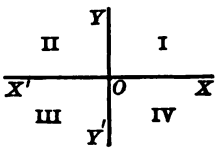


In the same way we may locate a chair on the floor with reference to the east and north wall, a spring in a field with reference to two fences meeting at right angles, or a point on the blackboard with reference to two lines perpendicular to each other.

Some of this work of locating points, given their distances from two lines intersecting at right angles, has already been done in connection with the study of the negative number (p. 23).

146. Axes. The two lines at right angles to each other, from which distances are measured in locating a point, are called *axes.*

The horizontal axis is called the *axis of* x and is lettered $X'X$, as in the figure at the right. The vertical axis is called the *axis of* y and is lettered YY'.



147. Origin. The intersection of the two axes is called the *origin.*

The origin is usually lettered O, as in this figure.

148. Quadrants. All that part of the plane above the axis of x and to the right of the axis of y is called the *first quadrant;* that above the axis of x and to the left of the axis of y, the *second quadrant;* that below the axis of x and to the left of the axis of y, the *third quadrant;* and that below the axis of x and to the right of the axis of y, the *fourth quadrant.*

149. Signs. Distances to the right of the axis of y or above the axis of x are considered *positive;* those to the left of the axis of y or below the axis of x are considered *negative.*

Thus we consider temperatures below zero as negative. Similarly, we might consider north latitude as positive and south latitude as negative, and east longitude as positive and west longitude as negative. Whatever direction is considered positive, the opposite direction is considered negative.

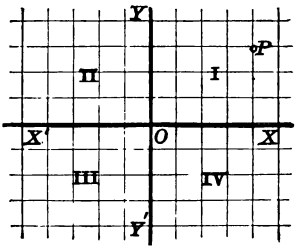
150. Designation of Points. In locating a point it is customary to state first the distance to the right or left of the vertical axis, parallel to the axis of x, and then the distance up or down from the horizontal axis, parallel to the axis of y.

Thus the point 2, 3 is in the first quadrant, 2 units to the right of the axis of y and 3 units above the axis of x.

Similarly, -4, 5 is in the second quadrant, 4 units to the left of the axis of y and 5 units above the axis of x.

A point $-a$, $-b$ is evidently in the third quadrant, a units to the left of the axis of y and b units below the axis of x. A point a, $-b$ is in the fourth quadrant, a units to the right of the axis of y and b units below the axis of x.

151. Coördinates. The distances of a point to the right or left of the axis of y, and above or below the axis of x, are called the *coördinates* of the point.



Thus the point P is in the first quadrant. It is designated as the point (4, 3), and its coördinates are 4 and 3. A point (− 4, 3) would be represented in the second quadrant, and so on.

152. Abscissa. The distance of a point from the vertical axis, measured on or parallel to the axis of x, is called the *abscissa* of the point.

Thus the abscissa of P is 4.

The abscissa of a point in the second quadrant is negative. The abscissa of a point in the third quadrant is also negative. The abscissa of a point in the fourth quadrant is positive.

153. Ordinate. The distance of a point from the horizontal axis, measured on or parallel to the axis of y, is called the *ordinate* of the point.

The ordinate of P is 3.

The ordinate of a point in the first or second quadrant is positive. The ordinate of a point in the third or fourth quadrant is negative.

154. Plotting a Point. The representing of a point by means of its coördinates is called the *plotting* of the point.

To plot the point (− 2, − 3), take the abscissa − 2 and draw its ordinate − 3. The point is, therefore, in the third quadrant.

155. Coördinate Paper. Paper ruled in squares for convenience in plotting points is called *coördinate paper.*

Coördinate paper will be found of use in the representation of equations and in drawing many of the figures used in geometry.

Such paper can be purchased at stationers. It is also easily made by the student. Points may be plotted by the aid of a ruler without the use of coördinate paper.

Coördinate paper is also known as *cross-ruled paper*, *cross-section paper*, or *squared paper.*

Exercise 98. Plotting Points

Examples 1 to 6, oral — Examples 7 to 16, written

1. In what quadrant is the point (2, 5)?

2. In what quadrants are the points (− 2, − 6) and (4, − 2)?

3. Where is the point (0, 0)? (5, 0)? (0, 5)? (− 5, 0)?

4. What is the distance from (4, 0) to (− 4, 0)?

5. In what quadrant is (2, 7)? (− 2, 7)? (2, − 7)?

6. What is the distance from (4, 3) to (− 4, 3)? from (4, 3) to (4, − 3)? from (0, 0) to (0, 7)? from (− 2, 5) to (− 2, − 5)?

7. Plot the points (1, 3), (7, 6), (5, 2), (9, 1), (1, 9), (5, 5).

8. Plot the points (− 2, 3), (− 4, 6), (− 6, 4), (− 5, 2).

9. Plot the points (− 2, − 3), (− 4, − 6), (− 5, − 2), (− 6, − 9), (− 7, − 7).

10. Plot the points (2, − 3), (3, − 2), (4, − 6), (6, − 4), (7, − 7), (− 7, 9), (0, − 4), ($1\frac{1}{2}$, $2\frac{1}{4}$), (− 0.5, − 3).

11. Plot the points (4, 4), (2, 1), (4, 1), (2, 4), (− 4, − 4), (− 2, − 1), (− 4, − 1), (− 2, − 4).

12. Plot the points (3, 5), (2, 3), (1, 5), (2, 6), (5, 3), (2, 0), (− 3, − 5), (− 2, − 3), (− 1, − 5).

13. Plot the points (1, 4), (4, 5), (4, − 4), (1, 1), (1, − 1), (1, 3), (− 4, − 5), (− 1, − 1), (− 1, 4), (0, 4).

14. What letter is formed by joining (1, 1) and (1, 4), (1, 4) and (3, 1), (3, 1) and (5, 4), (5, 4) and (5, 1)?

15. Join in succession the points (0, 4), (1, 1), (2, 3), (3, 1), and (4, 4). What letter is formed?

16. Join (0, 0) and (3, 5), (3, 5) and (6, 0), (1.2, 2) and (4.8, 2), (7, 0) and (12, 5), (7, 5) and (12, 0), (13, 0) and (13, 5), (18, 4) and (17, 5), (17, 5) and (15, 5), (15, 5) and (14, 4), (14, 4) and (14, 3), (14, 3) and (18, 2), (18, 2) and (18, 1), (18, 1) and (17, 0), (17, 0) and (15, 0), (15, 0) and (14, 1). What word is spelled by these lines?

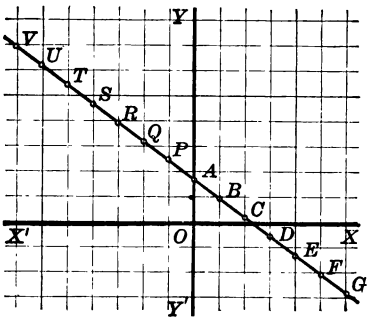
156. Plotting an Equation. Not only is it possible to plot a point, but it is also possible to plot an equation. For example, consider the equation $3x + 4y = 7$.

Solving for y we have $y = \frac{7 - 3x}{4}$. That is, y is a function of x (§ 45).

Evidently for any value that we may give to x we may find a corresponding value of y.

Thus if

$$x = 1, y = \frac{7 - 3 \times 1}{4} = 1.$$



We may now plot the point (1, 1) and we have one point that lies on the line. This is represented by B on the diagram.

Let $x =$	0	1	2	3	4	5	6
Then $y =$	$1\frac{3}{4}$	1	$\frac{1}{4}$	$-\frac{1}{2}$	$-1\frac{1}{4}$	-2	$-2\frac{3}{4}$
Point $=$	A	B	C	D	E	F	G

Let $x =$	-1	-2	-3	-4	-5	-6	-7
Then $y =$	$2\frac{1}{2}$	$3\frac{1}{4}$	4	$4\frac{3}{4}$	$5\frac{1}{2}$	$6\frac{1}{4}$	7
Point $=$	P	Q	R	S	T	U	V

We might also take fractional values of x and find corresponding values of y and corresponding points. All such points would be found to lie on a straight line, as shown in the diagram.

157. Equation of the First Degree. An equation of the first degree in two unknown quantities represents a straight line.

158. Graph. The line representing an equation is called the *graph* of the equation.

To *plot*, or *graph*, an equation means to draw the graph of the equation.

Exercise 99. Graphs of Equations

Examples 1 to 3, oral — Examples 4 to 22, written

1. In the equation $y = 4 - x$, find the value of y when $x = 1$; when $x = 2$; when $x = 4$; when $x = 7$; when $x = 0$; when $x = -4$.

2. In the equation $x = y - 5$, find the value of x when $y = 0$; when $y = 1$; when $y = 4$; when $y = 5$; when $y = 10$; when $y = -3$.

3. In the equation $x - y = 7$, what does x equal when $y = 7$? What does y equal when $x = 7$?

4. Make a table like that on page 173, giving values of y for fourteen values of x, in the equation $5x - 2y = 7$.

5. Make a similar table for the equation $3x - 4y = 9$.

6. Make a similar table for the equation $2x = 3y$.

7. Plot the equation given in Ex. 4.

8. Plot the equation given in Ex. 5.

9. Plot the equation given in Ex. 6.

Plot the following equations, fixing five points in each:

10. $x + 2y = 5$.
11. $x - 2y = 5$.
12. $3x + 4y = 7$.
13. $3x - 4y = 7$.
14. $5x + 7y = 12$.
15. $7x = 8y$.
16. $8x = 7y$.
17. $x = y$.
18. $2.7x + 1.2y = 3.9$.
19. $4.1x - 2.3y = 1.9$.

20. Plot the equation of which the graph cuts the axis of x at 5 and the axis of y at 3.

21. Plot the equation of which the graph passes through the points (0, 0) and (5, 5). At what angle does it seem to cut the axis of x?

22. Plot the equation of which the graph passes through the points (0, 0) and (− 5, 5). At what angle does it seem to cut the axis of y?

159. Linear Equations. Because an equation of the first degree involving two unknown quantities has for its graph a straight line, all equations of the first degree are known as *linear equations.*

160. Variable. A quantity that, under the conditions of a problem, may take different values is called a *variable.*

In the equation $y = 4x - 2$ we may give to x various values and fron these we may find corresponding values for y. This equation has therefore two variables, x and y.

161. Constant. A quantity that, under the conditions of a problem, has a fixed value is called a *constant.*

Variables are usually represented by the last letters of the alphabet, and constants by the first letters of the alphabet or by numerals.

162. Special Directions. Since two points exactly fix the position of a straight line, we need fix only two points in plotting a linear equation involving two variables. If we let $y = 0$ we have for the value of x the distance to the point where the graph cuts the axis of x. If we let $x = 0$ we have for the value of y the distance to the point where the graph cuts the axis of y.

In the equation

$$3x - 5y = 15,$$

if $x = 0$, $y = -3$,

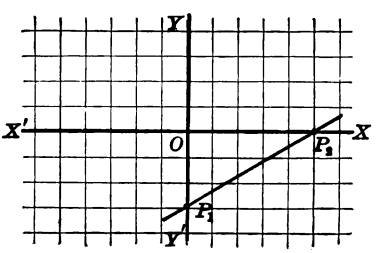
and if $y = 0$, $x = 5$.

Here

$(0, -3)$ is the point P_1,

and $(5, 0)$ is the point P_2.

Therefore the required graph is P_1P_2.

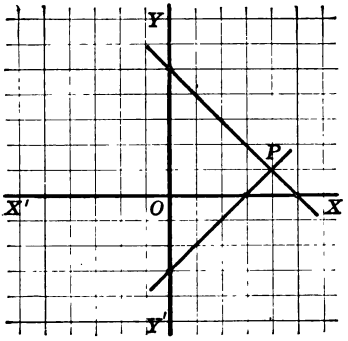


Any other two points may be taken to fix the line.

The equation $x = 5$ is equivalent to $x + 0 \cdot y = 5$, and hence, whatever value y has, x always equals 5. The graph is therefore parallel to the axis of y, 5 units to the right of the axis.

Similarly, the equation $x = -2$ represents a line 2 units to the left of the axis of y, and parallel to it; and $y = 4$ represents a line 4 units above the axis of x, and parallel to it.

163. Two Linear Equations. Two linear equations involving two variables are represented by two straight lines. These lines can intersect in only one point. Therefore, in general,



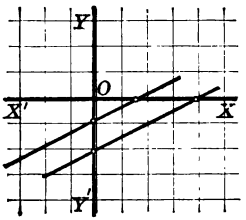
The graphs of two linear equations involving two variables have only one point in common;

Two linear equations involving two variables have only one pair of values of the variables in common.

Thus the graphs of $x + y = 5$ and $x - y = 3$ intersect at P. The coördinates of P are 4 and 1. Hence $x = 4$, $y = 1$.

There are an infinite number of points on each graph, but there is only one point on both graphs. Similarly, there are an infinite number of values of x and y that will satisfy each equation, but there is only one value that will satisfy both equations.

164. Inconsistent Equations. If we plot the equations $x - 2y = 4$ and $3x - 6y = 5$, we shall have two parallel lines. Such lines have no point in common. Considering the equations, we see that the second one reduces to $x - 2y = 1\frac{2}{3}$. The equations are therefore *inconsistent*, since $x - 2y$ cannot equal both 4 and $1\frac{2}{3}$.



165. Equivalent Equations. If we plot the equations $2x + 4y = 5$ and $x + 2y = 2\frac{1}{2}$, we shall find that one graph coincides with the other. They have an infinite number of points in common, and therefore an infinite number of values of x and y satisfy the equations, every pair of roots of either being a pair of roots of the other. The equations are therefore *equivalent*.

Exercise 100. Graphs of Linear Equations

Examples 1 to 4, oral — Examples 5 to 36, written

1. What is the nature of the graph of $x = 2$? of $y = 2$?
2. What is the nature of the graph of $y = 4$? of $y = -4$?
3. State two points on the graph of $x + 4y = 8$.
4. State two points through which the graph of $x = y$ passes.

Plot the following equations, and solve by measuring the coördinates of the point of intersection of the graphs:

5. $x + 4y = 11$
   $2x - y = 4$
6. $2x + 3y = 19$
   $7x - 2y = 4$
7. $x + 5y = -3$
   $2x - 3y = 20$
8. $2x - 9y = 23$
   $5x + y = -13$
9. $x + 5y = 0$
   $3x + 9y = -6$
10. $7x + 2y = 14$
    $5x - 3y = -21$
11. $2x - 3y = 7$
    $5x - 7y = 14$
12. $6x - 3y = 15$
    $2x + 7y = 45$

13. Show by graphs that the equations $x + 4y = 6$ and $0.5x + 2y = 4$ are inconsistent.

14. Show by graphs that the equations $0.2x - 0.5y = 6$ and $x - 2.5y = 30$ are indeterminate.

15. Show by graphs that the three equations $x + y = 6$, $2x - y = 0$, and $5x + 3y = 22$ have a common root. Find the root.

16. Show by graphs that the three equations $x + y = 5$, $2x - 3y = 20$, and $3x + y = 2$ have no root common to all three.

17. If $x + 5y = 2.1$, y equals what function of x? x equals what function of y? If $2x - y = 2$, y equals what function of x? x equals what function of y?

18. Taking $y = f(x)$ and $y = F(x)$ as found in Ex. 17, plot these two equations, and solve by measuring the coördinates.

Plot the following equations, and determine what sets admit of solution and what do not:

19. $3x + 6y = 7$
$2x - 4y = 5$

20. $3x + 6y = 7$
$2x + 4y = 5$

21. $3x + 6y = 7$
$2x + 4y = 4\frac{2}{3}$

22. $3x - 6y = 7$
$2x + 4y = 5$

23. $3x - 6y = 7$
$2x - 4y = 5$

24. $2x + 7y = 27$
$5x - 2y = 9$

25. $x + y = 9$
$x - y = 1$

26. $x + 7y = 14$
$7x - y = 48$

27. $3x + 8y = 12$
$4.5x + 12y = 16$

28. $5x - 7y = 6$
$6x - 8.4y = 8$

29. $2x - 5y = 3$
$5x - 2y = 3$

30. $5x + 12.8y = 35$
$3\frac{1}{7}x - 4.8y = 22$

Determine by means of graphs whether one or more of the following sets of equations have a root common to all three equations of the set:

31. $x + 2y = 8$
$y - x = 1$
$4x + y = 6$

32. $5x - y = 7$
$x + 7y = 23$
$2x + y = 5$

33. $4x + y = 21$
$4y + x = 9$
$4x + 4y = 12$

34. $x + y = 9$
$x - y = 5$
$2x + 3y = 20$

35. When we consider the graphs of two linear equations containing only x and y, how does it appear that such equations are usually simultaneous? Draw the graphs of two equations to illustrate this fact.

36. When we consider the graphs of three linear equations containing only x and y, how does it appear that such equations are usually not simultaneous? Draw the graphs of three equations to illustrate this fact.

CHAPTER XV

POWERS AND ROOTS

166. Power. The product of several equal factors is called a *power* of the factor.

167. Laws of Exponents. The laws of exponents are these:

1. *Products:* $a^m a^n = a^{m+n}$. § 60

2. *Quotients:* $a^m \div a^n = a^{m-n}$. § 69

3. *Powers of products:* $(ab)^m = a^m b^m$.

For $(ab)^m = ab \cdot ab \cdot ab \cdots m$ times
$= (aaa \cdots \text{to } m \text{ factors})\,(bbb \cdots \text{to } m \text{ factors})$
$= a^m b^m$.

Conversely, $a^m b^m = (ab)^m$.

Similarly, $(abc)^m = a^m b^m c^m$, and so on for any number of factors.

4. *Powers of quotients:* $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.

For $\left(\frac{a}{b}\right)^m = \frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b} \cdots \text{to } m \text{ factors}$
$= \frac{aaa \cdots \text{to } m \text{ factors}}{bbb \cdots \text{to } m \text{ factors}}$
$= \frac{a^m}{b^m}$.

Conversely, $\frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m$.

5. *Powers of powers:* $(a^m)^n = a^{mn}$.

For $(a^m)^n = (aaa \cdots \text{to } m \text{ factors})^n$
$= a^n a^n a^n \cdots \text{to } m \text{ factors, by Law 3}$
$= a^{n+n+n \cdots \text{to } m \text{ terms}}$, by Law 1
$= a^{mn}$.

168. Law of Signs. From the law of signs in multiplication (§ 61) we have the following law:

Even powers of a positive or a negative number are positive; odd powers have the same sign as the number itself.

Thus $(+3)^2 = +9$, $(+a)^2 = +a^2$, $(\pm a)^4 = +a^4$,
and $(-3)^2 = +9$, $(-a)^2 = +a^2$, $(\pm a)^{10} = +a^{10}$.
Likewise, $(+2)^3 = +8$, $(+a)^3 = +a^3$, $(+a)^5 = +a^5$,
and $(-2)^3 = -8$, $(-a)^3 = -a^3$, $(-a)^5 = -a^5$.

Exercise 101. Powers of Monomials

Examples 1 to 6, oral — Examples 7 to 35, written

1. State the product of $a^2 \cdot a^3$; of $a^4 \cdot a^7$; of $x^2 \cdot x^3 \cdot x^4$.
2. State the product of $a^p \cdot a^q$; of $a^m \cdot a$; of $a^p \cdot a^q \cdot a^r$.
3. State the quotient of $a^5 \div a^2$; of $3^3 \div 3^2$; of $a^p \div a^q$.
4. Express without the parentheses: $(ab)^p$; $(abcd)^m$; $(xyz)^3$.
5. Express without the parentheses: $(a^p)^q$; $(a^q)^p$; $(a^{mn})^p$; $(a^m b^n)^p$; $(a^x b^y c^z)^2$; $(a^w b^x c^y d^z)^4$.
6. Express without the parentheses: $\left(\frac{2}{3}\right)^2$; $\left(\frac{a}{b}\right)^2$; $\left(\frac{a^m}{b^m}\right)^p$; $\left(\frac{a^m}{b^n}\right)^3$.

Express the following without the parentheses:

7. $(a^2)^5$.	13. $(-a^2)^4$.	19. $(a^m b^n)^p$.	25. $(ab^2c^3d^4)^5$.
8. $(a^5)^2$.	14. $(-2^3)^2$.	20. $(a^5b^7)^m$.	26. $(4a^2b^3c)^3$.
9. $(2^3)^2$.	15. $(-ab)^9$.	21. $(-a^9b^9)^9$.	27. $(-2a^5b^5)^4$.
10. $(3^2)^3$.	16. $(-a^2b^3)^3$.	22. $(-a^3b^3)^3$.	28. $(-3a^7b^9)^3$.
11. $(a^2x^3)^7$.	17. $(-a^m b^n)^2$.	23. $(-a^m b^n)^{15}$.	29. $(-2a^9b^5)^6$.
12. $(a^5b^9)^{13}$.	18. $(-a^m b^n)^3$.	24. $(-a^m b^n)^{16}$.	30. $(-2x^7y^9)^5$.

31. Is $2n$ even or odd? Is $2n+1$ even or odd? What is the sign of $(-a)^{2n}$? of $(-a)^{2n+1}$?

Express the following without the parentheses:

32. $\left(\frac{a^7}{b^9}\right)^6$. 33. $\left(-\frac{p^8}{q^{10}}\right)^5$. 34. $\left(-9\frac{x^5}{y^7}\right)^2$. 35. $\left(-\frac{a^m}{b^n}\right)^{2n+1}$.

169. Root. Any one of the equal factors whose product is a given number is called a *root* of the number.

The term is already familiar from arithmetic and has therefore been used whenever necessary in the previous work (§ 7).

The root is called a square root, cube root, fourth root, or rth root, according as it is one of two, three, four, or r equal factors.

The word *root* is used in two different senses in algebra. The other use is to represent the value of the unknown quantity in an equation. Thus the square root of 25 is 5, and the root of the equation $x - 7 = 23$ is 30.

170. Radical Sign. The common symbol for root $(\sqrt{\ })$ is called the *radical sign.*

Thus, $\sqrt[4]{625} = 5$, because $625 = 5 \cdot 5 \cdot 5 \cdot 5$. In this case 4 is called the *index* of the root. In $\sqrt[n]{a - b}$, n is the index of the root. In the case of a square root, such as $\sqrt{4}$, the index is not written.

171. Imaginary Number. Since any even power of a positive or a negative number is positive (§ 168), therefore an even root of a negative number cannot be positive and cannot be negative.

We can merely indicate such roots thus: $\sqrt{-1}$, $\sqrt{-3}$, $\sqrt[4]{-2}$. Such numbers are treated later in our work.

An indicated square root of a negative number is called an *imaginary number.*

172. Real Number. A number that does not contain an imaginary number is called a *real number.*

A real number may be an integer (whole number) like 3 or -7, a fraction like $\frac{2}{3}$, $\frac{5}{2}$, $-\frac{5}{8}$, or 0.7, a mixed number like $1\frac{1}{2}$ or -2.75, or such a number as $\sqrt{3}$ or π; but $2 + 3\sqrt{-5}$ is not a real number.

173. Rational Number. An integer or the quotient of two integers is called a *rational number.*

For example, 3, -7, $\frac{2}{3}$, $\frac{5}{2}$, 1.25, and -0.75 are rational numbers.

174. Irrational Number. A real number that is not rational is called an *irrational number.*

For example, $\sqrt{2}$ and $\sqrt{\frac{1}{2}}$ are both irrational. Similarly, numbers like $2\sqrt{3}$ and $\sqrt[3]{\frac{1}{2}}$ are irrational.

175. Signs of Roots. Since $(+2)\cdot(+2)=4$, and $(-2)(-2)=4$, we see that 4 has two square roots, $+2$ and -2. For the same reason it is evident that

Every positive number has two real square roots which have the same absolute values but have opposite signs.

Similarly, every positive number has at least two real fourth roots, two real sixth roots, and so on for any roots indicated by an even number.

For example, 9 has two real square roots, $+3$ and -3; and 16 has two fourth roots, $+2$ and -2, since $(+2)^4=16$, and $(-2)^4=16$. Imaginary roots are discussed later.

As a matter of fact every number has three cube roots, four fourth roots, five fifth roots, and so on. Some of these roots are imaginary, but we do not need to consider such roots at the present time.

Furthermore, since $(+2)^3=+8$, and $(-2)^3=-8$, we see that $\sqrt[3]{+8}=+2$, and $\sqrt[3]{-8}=-2$. For the same reason it is evident that

An odd root of a number has the same sign as the number itself.

For example, because $(-2)^5=-32$, therefore $\sqrt[5]{-32}=-2$, while $\sqrt[5]{+32}=+2$.

176. Principal Root. The one real root of a number, if it has but one, or its real positive root, if it has two real roots, is called the *principal root* of the number.

Thus the principal square root of 4 is $+2$, although there is another root, -2; and the principal cube root of -27 is -3, although it will later be shown that there are two imaginary roots.

177. Radical Sign and Principal Root. It is to be understood in algebra that *the radical sign indicates the principal root.*

Thus $\sqrt{4}=2$, and it is incorrect to write $\sqrt{4}=-2$, although 4 has two square roots. If we wish to indicate the negative root, we write $-\sqrt{4}=-2$, and if we wish to indicate both roots, we write $\pm\sqrt{4}=\pm 2$.

Therefore if we have to simplify $\sqrt{4}+\sqrt{9}+\sqrt{16}$, we have $2+3+4=9$, and not $\pm 2\pm 3\pm 4$, which has several possible values.

Similarly, if we have the equation $x^2=5$ we should write $x=\pm\sqrt{5}$ and not merely $x=\sqrt{5}$.

178. Square Root of a Polynomial. If we square $a + b$ the result is, as shown in § 87, $a^2 + 2ab + b^2$. In extracting the square root of $a^2 + 2ab + b^2$ we reverse the process of squaring, thus:

The given square =	$a^2 + 2ab + b^2$	$\vert$	$a + b$ = Root
The square of a =	a^2		
Trial divisor =	$2a$	$\vert$	$2ab + b^2$ = First remainder
Complete divisor =	$2a + b$	$\vert$	$2ab + b^2 = b(2a + b)$

The first term of the root is evidently a, because the square of a is a^2.

Since in squaring a binomial we have the square of the first term plus twice the product of the first and second terms, etc., we have in $2ab$ twice the product of a and the second term. We therefore divide by $2a$ to find the second term. Since $2ab \div 2a$ equals b, b is the second term.

In squaring $a + b$ we have $a^2 + b(2a + b)$. We therefore add b to $2a$, and multiply $2a + b$ by b, thus completing the square of $a + b$.

The names *trial divisor* and *complete divisor* are used as above shown, being convenient names brought into algebra from arithmetic.

1. Find the square root of $9x^2 - 30xy^2 + 25y^4$.

The given square =	$9x^2 - 30xy^2 + 25y^4$	$\vert$	$3x - 5y^2$ = Root
The square of $3x$ =	$9x^2$		
Trial divisor =	$6x$	$\vert$	$-30xy^2 + 25y^4$
Complete divisor =	$6x - 5y^2$	$\vert$	$-30xy^2 + 25y^4$

The student should compare this, step by step, with the example explained above, answering the following questions:

How is the first term of the root found?

Why is $6x$ taken as the divisor?

How is the second term found?

Why is $-5y^2$ added to $6x$?

Why is $6x - 5y^2$ multiplied by $-5y^2$?

2. Find the square root of $81x^2y^2 - 90xy + 25$.

	$81x^2y^2 - 90xy + 25$	$\vert$	$9xy - 5$
	$81x^2y^2$		
$18xy$		$\vert$	$-90xy + 25$
$18xy - 5$		$\vert$	$-90xy + 25$

This example is merely illustrative. In such a simple case we would ordinarily find the square root by inspection.

3. Find the square root of $4x^4 - 12x^2y + 9y^2 + 4x^2z^3 - 6yz^3 + z^6$.

$$
\begin{array}{l|l}
4x^4 - 12x^2y + 9y^2 + 4x^2z^3 - 6yz^3 + z^6 & 2x^2 - 3y + z^3 \\
4x^4 & \\
\hline
4x^2 & -12x^2y + 9y^2 \\
4x^2 - 3y & -12x^2y + 9y^2 \\
\hline
4x^2 - 6y & 4x^2z^3 - 6yz^3 + z^6 \\
4x^2 - 6y + z^3 & 4x^2z^3 - 6yz^3 + z^6
\end{array}
$$

The first term of the root is $2x^2$.

Subtracting the square of $2x^2$, which is $4x^4$, the remainder contains twice the first term times the second, plus the square of the second. Dividing by $4x^2$, which is twice $2x^2$, the second term is $-3y$.

Proceeding as before and subtracting, we may consider the second remainder as twice the product of $(2x^2 - 3y)$ by the next term, considering the binomial $2x^2 - 3y$ as the first term.

Dividing by twice $2x^2 - 3y$, or by $4x^2 - 6y$, we find the next term to be z^3.

Proceeding as before and subtracting, there is no remainder. Hence the square root is $2x^2 - 3y + z^3$.

We may check this result by squaring it, or we may let $x = 2$, $y = 1$, $z = 1$ (or any other convenient values). Then we have $6^2 = 36$.

If the terms of the given polynomial are not arranged according to the ascending or descending powers of some letter, make this arrangement before beginning the work.

179. Extracting the Square Root. Therefore, in extracting the square root of a polynomial,

Arrange the terms according to the powers of some letter.

Find the principal square root of the first term, and subtract its square from the polynomial.

Divide the first term of the remainder by twice the first term of the root, and write the quotient as the second term of the root.

Multiply the sum of twice the first term and the second term of the root by the second term, thus completing the square of the sum of the first two terms, and subtract this product.

Consider what has now been found as the first part of the root, and proceed as before.

Exercise 102. Square Root of a Polynomial

Examples 1 to 8, oral — Examples 9 to 27, written

1. What is the square root of $9x^2 + 6x + 1$?
2. What is the square root of $16x^2 - 8x + 1$?
3. What are the two square roots of $x^2y^2 - 6xy + 9$?

For both square roots, inclose the root in parentheses preceded by the $\pm$ sign.

Find the square roots of the following:

4. $m^2 - 8m + 16$.
5. $p^4 - 4p^2q^2 + 4q^4$.
6. $\frac{a^2}{16} + \frac{2a}{4} + 1$.
7. $1 + \frac{6}{m} + \frac{9}{m^2}$.
8. $4x^2 + 4 \cdot \frac{x}{y} + \frac{1}{y^2}$.
9. $16m^2 + 24mn + 9n^2$.
10. $x^4 + 16x^2y^2 + 64y^4$.
11. $\frac{1}{a^2} - \frac{2}{ay} + \frac{1}{y^2}$.
12. $81a^2 - 126 + \frac{49}{a^2}$.
13. $4x^4 + 20x^2 - 3 - \frac{70}{x^2} + \frac{49}{x^4}$.
14. $4a^2 - 12a + \frac{4}{a} + 9 - \frac{6}{a^2} + \frac{1}{a^4}$.
15. $p^4 - 2p^3 + 3p^2 - 2p + 1$.
16. $a^4 + 2a^3 + 5a^2 + 4a + 4$.
17. $16x^4 + 16x^3 + 20x^2 + 8x + 4$.
18. $a^2 + 2ay + y^2 + 2a + 2y + 1$.
19. $b^2 + 2by + y^2 + 2bz + 2yz + z^2$.
20. $p^4 - 6p^3 + 17p^2 - 24p + 16$.
21. $9a^4 - 12a^3 + 34a^2 - 20a + 25$.
22. $4p^6 - 12p^5 - 7p^4 + 44p^3 - 14p^2 - 40p + 25$.
23. $25a^6 - 30a^5y + 9a^4y^2 + 10a^3y^3 - 6a^2y^5 + y^4$.
24. $29p^4b^2 + 16b^6 - 12p^5b - 46p^3b^3 + 4p^6 - 40pb^5 + 49p^2b^4$.
25. $16a^2 + 8ay + y^2 + 8az + 2yz + z^2 + 8a + 2y + 2z + 1$.
26. $b^2 + 8by + 16y^2 + 2bz + 8yz + z^2 + 2b + 8y + 2z + 1$.
27. $x^2 + 2xy + y^2 + 8xz + 8yz + 16z^2 - 2x - 2y - 8z + 1$.

180. Arithmetical Square Root. The first step in extracting the square root of a number is to separate the figures of the number into groups of two figures each, called *periods*.

Since $1 = 1^2$, $100 = 10^2$, $10{,}000 = 100^2$, and so on, it is evident that the square root of any number between 1 and 100 lies between 1 and 10; of any number between 100 and 10,000 lies between 10 and 100. In other words, the square root of any integral number expressed by *one* or *two* figures is a number of *one* figure; expressed by *three* or *four* figures is a number of *two* figures; and so on.

Therefore if an integral number is separated into periods of two figures each, from the right to the left, the number of figures in the square root will be equal to the number of the periods of figures. The last period at the left may have two figures or one figure; for example, 22 09, and 7 89 04 81.

1. Find the square root of 2209.

We first recall that if t = tens and u = units, we have $(t + u)^2 = t^2 + 2tu + u^2$.

Separating into periods, we see that there will be two integral places in the root.

The first period, 22, contains t^2. And since the greatest square in 22 is 16, therefore $\sqrt{16}$, or 4, is the tens' figure of the root, or t.

Subtracting t^2, the remainder contains $2tu + u^2$. Therefore if we divide by $2t$ (that is, by 80, which is 2×4 tens), we shall find approximately u, the units. Dividing 609 by 80 (or 60 by 8), we have 7 as the units' figure.

```
 22 09 (47
 16
---
80  6 09
87  6 09
---
```

Since $2tu + u^2 = (2t + u)u$, that is, $2 \times 40 \times 7 + 7^2 = (2 \times 40 + 7) \times 7$, we add 7 to 80 and multiply the sum by 7. The product is 609, thus completing the square of 47.

2. Find the square root of 7,890,481.

When the third period, 04, is brought down, and the divisor, 560, formed, the next figure of the root is 0, because 560 is not contained in 504. Therefore 0 is placed in the root, another 0 is annexed to the divisor, and the next two figures, 81, are brought down.

```
  7 89 04 81 (2809
  4
---
40  3 89
48  3 84
---
5600    5 04 81
5609    5 04 81
---
```

The work then proceeds as before, the trial divisor now being 5600, which is twice 280 tens. The entire root is thus found to be 2809.

181. Square Roots of Decimals. If the square root of a number has decimal places, the number itself will have twice as many.

Thus if 0.25 is the square root of some number, the number will be 0.25^2, or 0.25×0.25, or 0.0625. Hence if a given number contains a decimal, we separate the number into periods of two figures each, beginning at the decimal point and proceeding toward the left for the integral part, and toward the right for the decimal. The last period of the decimal must have two figures, a zero being annexed if necessary.

Find the square root of 52.2729.

	52.27 29 (7.23
	49
140	3 27
142	2 84
1440	43 29
1443	43 29

We see at once that the root will have one integral place. Furthermore, if it is a perfect square it will have two decimal places, since the square of thousandths is millionths.

If a number is not a perfect square, zeros may be annexed and an approximate value of the root found to as many decimal places as desired.

Exercise 103. Arithmetical Square Roots

Examples 1 to 3, oral — Examples 4 to 25, written

1. State the squares of 11, 12, 13, 15, 20, 21, 25, 50.
2. State the square roots of 144, 169, 400, 900, 1600, 3600
3. What is the square root of 121? of 1.21? of 0.0121?

Extract the square root of:

4. 190,969.	9. 804,609.	14. 1036.84.	19. 3.9204.
5. 743,044.	10. 194,481.	15. 82.2649.	20. 462.25.
6. 401,956.	11. 173,056.	16. 0.063001.	21. 0.003969.
7. 758,641.	12. 174,724.	17. 1.5129.	22. 0.182329.
8. 117,649.	13. 509,796.	18. 2.6244.	23. 0.054756.

Find the side of a square whose area is:

24. 12,321 sq. ft.
25. 8046.09 sq. in.

182. Square Roots of Common Fractions. To find the square root of a common fraction we may extract the square roots of the numerator and denominator.

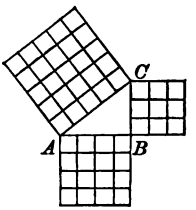
If the denominator is not a perfect square, it is better to multiply both terms of the fraction by a number that will make it such, or else to reduce to a decimal and then extract the square root. Thus, instead of saying $\sqrt{\frac{1}{2}} = \frac{\sqrt{1}}{\sqrt{2}} = \frac{1}{1.414+}$, which requires us to divide the numerator by a long divisor, it is better to say $\sqrt{\frac{1}{2}} = \sqrt{\frac{2}{4}} = \frac{\sqrt{2}}{2} = \frac{1}{2}\sqrt{2} = \frac{1}{2}$ of $1.414+ = 0.707+$, or else to extract the square root of 0.5.

183. Right Triangle. A triangle that contains a right angle is called a *right triangle*. It is sometimes called a *right-angled triangle*.

Certain definitions of this kind, familiar from the work in arithmetic, are reviewed from time to time because of their relation to the particular topic under discussion.

184. Hypotenuse. In a right triangle the side opposite the right angle is called the *hypotenuse*.

In the triangle ABC the hypotenuse is AC, the base is AB, and the perpendicular (or altitude) is BC.



185. Square on the Hypotenuse. By examining this figure we see that the square on AC equals the sum of the squares on AB and BC. It is proved in geometry that this is true for all right triangles.

Therefore, *in a right triangle the square of the hypotenuse equals the sum of the squares of the other two sides.*

The hypotenuse equals the square root of the sum of the squares of the other two sides.

The base or the perpendicular equals the square root of the difference of the squares of the hypotenuse and the other side.

This is one of the most important propositions of geometry. It is said to have been first proved in the sixth century B.C. by the Greek mathematician Pythagoras, a native of Samos, in the Ægean Sea.

Exercise 104. Arithmetical Square Roots

Examples 1 and 2, oral — Examples 3 to 39, written

1. Find the square root of $\frac{25}{144}$, $\frac{36}{121}$, $\frac{49}{81}$, $\frac{64}{121}$.

2. Find the square root of $\frac{121}{169}$, $\frac{441}{625}$, $\frac{100}{121}$, $\frac{121}{400}$, $\frac{144}{169}$.

Find the square root of:

3. $\frac{961}{1089}$. 4. $\frac{1369}{1681}$. 5. $\frac{841}{1849}$. 6. $\frac{2209}{2601}$. 7. $\frac{2809}{3025}$.

Find the square root, to two decimal places:

8. 2. 9. 11. 10. 30. 11. 125. 12. $\frac{5}{8}$.

Reduce to fractions with denominators that are perfect squares, and find the square root to two decimal places:

13. $\frac{1}{3}$. 14. $\frac{2}{3}$. 15. $\frac{2}{5}$. 16. $\frac{3}{8}$. 17. $\frac{7}{8}$.

Find the square root, to two decimal places, of:

18. 0.625. 19. 0.571. 20. 0.6. 21. 0.423. 22. 0.916.

Find the hypotenuse, given the other sides as follows:

23. 39 ft., 52 ft. 24. 21 ft., 72 ft. 25. 51 ft., 68 ft.

Also as follows, carrying the results to three decimals:

26. 82 ft., 35 ft. 27. 31 ft., 23 ft. 28. 27 ft., 43 ft.

Given the hypotenuse and one side, find the other side:

29. 10 ft., 6 ft. 30. 17 in., 15 in. 31. 25 in., 20 in.

Also as follows, carrying the results to three decimals:

32. 15 ft., 6 ft. 33. 18 ft., 12 ft. 34. 23 in., 12 in.

Find the diagonal of the square whose side is:

35. 20″. 36. 32″. 37. 45″. 38. 70″. 39. 75″.

186. Applications of Square Root. It is proved in geometry that

The corresponding lines of similar figures are to each other as the square roots of the areas of the figures.

That is, if one circle has 16 times the area of another, the radius is 4 times as long as that of the other.

The radius of a circle equals approximately the square root of the quotient of the area of the circle divided by 3.1416.

For example, if the area of a circle is 78.54 sq. in., the number of inches in the radius is $\sqrt{\frac{78.54}{3.1416}} = \sqrt{25} = 5$. Therefore the radius is 5 in.

In Exercise 105 carry the square roots to two decimal places.

Exercise 105. Applications of Square Root

Examples 1 to 3, oral — Examples 4 to 9, written

1. The areas of two squares are 121 sq. in. and 144 sq. in. respectively. What is the ratio of their sides? of their diagonals?

2. The area of one circle is 81 times that of another. What is the ratio of their diameters? of their circumferences?

3. A rectangular lot is worth \$1000. Find the value of a similar lot which has twice the front and twice the depth, at the same rate per square foot.

Find the radius of the circle whose area is:

4. 314.16 sq. in. 5. 113.0976 sq. in. 6. 250 sq. in.

7. What must be the diameter of a water main to have the area of a cross section 3 sq. ft.?

8. A horse tethered by a rope can graze over 1570.8 sq. ft. of ground. How long is the rope? If the rope were twice as long, over how much ground could he graze?

9. A tinsmith wishes to make some cylindrical gallon cans. They are to be 10 in. high. What must be the area of the base? What radius must he use to draw it? (1 gal. = 231 cu. in.)

187. The *n*th Power of the *n*th Root. Because $\sqrt{3}$ means one of the two equal factors of 3, $\sqrt{3} \cdot \sqrt{3}$ must equal 3. That is, $(\sqrt{3})^2 = 3$. By the same reasoning

The nth power of the nth root of a number equals the number itself.

That is, $$(\sqrt[n]{a})^n = a.$$

For a similar reason, *the nth root of the nth power of a number equals the number itself.*

That is, $$\sqrt[n]{a^n} = a.$$

188. The *n*th Root of a Product. Because it has already been proved that

$$ab = (\sqrt{a})^2(\sqrt{b})^2 \qquad \S\ 187$$

$$= (\sqrt{a} \cdot \sqrt{b})^2, \qquad \S\ 167$$

therefore $$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}. \qquad \text{Axiom } 5$$

By the same reasoning

The nth root of the product of two or more numbers equals the product of their nth roots.

That is, $$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}.$$

Reversing this statement, $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$. That is, $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$, $\sqrt[3]{4} \cdot \sqrt[3]{3} = \sqrt[3]{12}$, etc.

189. The *n*th Root of a Power. Since we know that

$$\sqrt[3]{a^6} = \sqrt[3]{a^3a^3} = aa = a^2, \qquad \S\ 188$$

and that $$\sqrt[n]{a^{2n}} = \sqrt[n]{a^n a^n} = aa = a^2,$$

Therefore, *the exponent of the nth root of any power of a number is found by dividing the exponent of the power by the index of the root.*

That is, $$\sqrt[3]{x^{12}} = x^4, \quad \sqrt[5]{n^{10}} = n^2.$$

Similarly, $$\sqrt[n]{a^{mn}} = a^m.$$

If the exponent is not exactly divisible by the index of the root, we proceed as follows:

$$\sqrt[5]{n^{11}m^{20}} = \sqrt[5]{n^{10}m^{20}n} = n^2m^4\sqrt[5]{n}.$$

190. Surd. If an indicated root of a positive rational number cannot be obtained exactly, the indicated root is called a *surd*.

For example, $\sqrt{2}$ and $\sqrt[3]{\frac{2}{3}}$ are surds, but any expression in which the radical sign enters is called a *radical*, such as $a + \sqrt{b}$.

191. Surds classified. Surds are classified according to the index of the root, being called *quadratic*, *cubic*, and *biquadratic*, according as this index is 2, 3, or 4.

We might have other names, as *quintic* (fifth root), *sextic* (sixth root), and so on, but it is not necessary to use these terms.

Surds are also classified as *mixed surds*, when there is a rational coefficient; and as *entire surds*, when there is no rational coefficient other than 1.

Thus $2\sqrt{3}$ is a mixed surd, and $\sqrt[3]{7}$ is an entire surd.

192. Reduction of Surds. The changing of the form of a surd without changing its value is called a *reduction* of the surd.

Thus it will be seen that $\sqrt{4a} = 2\sqrt{a}$, and in this case we have reduced $\sqrt{4a}$ to another form.

193. Surds in Simplest Form. When the quantity under the radical sign is integral and as small as possible, a surd is said to be in its *simplest form*.

That is, a number left under the radical sign must be as small as possible, a letter must have as small an exponent as possible, and there must be no fraction left under the radical sign.

For example, $\sqrt{4a}$ is not in its simplest form, for we can extract the square root of 4, nor is $\sqrt[3]{2a^3}$ in its simplest form.

194. Illustrative Problems. Consider the following problems:

1. Simplify $\sqrt[3]{5^3}$, $(\sqrt[3]{5})^3$, and $\sqrt[n]{5^n}$.

$\sqrt[3]{5^3} = 5$, $(\sqrt[3]{5})^3 = 5$, and $\sqrt[n]{5^n} = 5$, by § 187.

2. Simplify $\sqrt{9a^2b^8}$, $\sqrt[3]{27a^6b^{12}}$, and $\sqrt[5]{32x^{10}}$.

$\sqrt{9a^2b^8} = 3ab^4$, $\sqrt[3]{27a^6b^{12}} = 3a^2b^4$, and $\sqrt[5]{32x^{10}} = 2x^2$, by §§ 188 and 189.

3. Simplify $\sqrt[5]{32a^6x^{10}}$.

$\sqrt[5]{32a^6x^{10}} = \sqrt[5]{2^5a^5x^{10}a} = 2ax^2\sqrt[5]{a}$, by §§ 188 and 189.

Exercise 106. Radicals

Examples 1 to 4, oral — Examples 5 to 36, written

1. Simplify $\sqrt[3]{a^3}$; $\sqrt[4]{x^4}$; $(\sqrt[5]{n})^5$; $\sqrt[5]{n^5}$; $\sqrt[7]{a^7b^7c^7}$.

2. Simplify $\sqrt{16\,a^2b^{10}}$; $\sqrt[3]{8\,a^6b^{15}}$; $\sqrt[4]{16\,a^{12}b^{20}}$; $\sqrt[n]{a^{2n}b^{mn}}$.

3. Simplify $\sqrt{4\,a^2b}$; $\sqrt[3]{27\,a^6b^2}$; $\sqrt[4]{81\,x^{12}y^5}$; $\sqrt[5]{32\,x^6y^8}$.

4. Simplify $\sqrt{(a+b)^2(a-b)}$; $\sqrt[4]{(a-b)^2}$; $\sqrt[10]{(x+y)^5}$.

Simplify the following:

5. $\sqrt{196\,a^{12}b^{12}c^{12}}$.

6. $\sqrt{256\,a^{10}b^{16}c^{18}}$.

7. $\sqrt{289\,a^4b^5c^7}$.

8. $\sqrt{361\,x^5y^6z^9}$.

9. $\sqrt{529\,m^2n^8p^9q^{11}}$.

10. $\sqrt[3]{216\,x^3y^3(x+y)^3}$.

11. $\sqrt[3]{27(a+b)^3(c+d)^3}$.

12. $\sqrt[3]{125(a+b+c)^4}$.

13. $\sqrt{1849(a+b)^2(a-b)^2}$.

14. $\sqrt[3]{1331(a-b)^4(c-d)^3}$.

15. $\sqrt[4]{625\,x^5y^5(x-y)^4}$.

16. $\sqrt[4]{(a-b)^4(b-c)^4(c-a)^4}$.

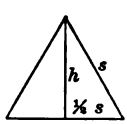
17. $\sqrt[3]{a^4+3\,a^3b+3\,a^2b^2+ab^3}$.

18. $\sqrt[5]{1024\,a^5b^5x^6y^5}$.

19. $\sqrt[5]{ab(a+b)^5(x+y)^5}$.

20. $\sqrt{(a^3+2\,a^2b+ab^2)(a^2+ab)}$

21. If the side of an equilateral triangle is 2 in., what is the altitude?



Since h is perpendicular to and bisects the base, therefore $h^2 + 1^2 = 2^2$, or $h^2 = 3$. Therefore $h = \sqrt{3}$.

22. If the side of an equilateral triangle is s, what is the altitude?

As before, h cuts off a right triangle with base $\frac{1}{2}s$ and hypotenuse s.

Hence $$h^2 + \left(\frac{s}{2}\right)^2 = s^2,$$

or $$h^2 = s^2 - \frac{s^2}{4} = \frac{3\,s^2}{4},$$

and $$h = \sqrt{\frac{3\,s^2}{4}} = \frac{s}{2}\sqrt{3}.$$

23. If the side of an equilateral triangle is $2\,m$, what is the altitude? the area?

24. If the side of an equilateral triangle is s, what is the area?

25. If the altitude of an equilateral triangle is a, what is the side? the area?

26. If the altitude of an equilateral triangle is 3, what is the side? the area?

27. If the area, a, of an equilateral triangle is $\frac{1}{4}s^2\sqrt{3}$, what is the side? the altitude?

28. If the area of an equilateral triangle is $9\sqrt{3}$ sq. in., what is the side? the altitude?

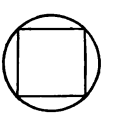
29. If the area of an equilateral triangle is 81 sq. in., find the length of the side to three decimal places.

30. If the area of an equilateral triangle is n sq. in., what is the side? the altitude?

31. From the formula for the area of a circle, $a = \pi r^2$, find the value of r^2. Knowing that $\frac{1}{\pi} = 0.3183$, nearly, find the value of r.

32. If the area of a circle is 50 sq. in., find to three decimal places the value of r.

33. Find the radius of the circle whose area is n sq. in. (Use the value of the reciprocal of π given in Ex. 31.)



34. Find the area of a square inscribed in a circle whose diameter is 10 in.

35. Find to two decimal places the area of a square inscribed in a circle whose circumference is $6\frac{1}{2}$ in.; in a circle whose circumference is 21 in.

36. How many square inches are there in the surface of the largest bar, having a square cross section, that can be cut from a cylinder 6 ft. long and 3 in. in diameter? Find the result to three decimal places.

195. Reduction of a Mixed Surd to an Entire Surd. Since we have found that $a^2 = \sqrt[3]{(a^2)^3} = \sqrt[3]{a^6}$,

we see that $a^2\sqrt[3]{b} = \sqrt[3]{a^6} \cdot \sqrt[3]{b} = \sqrt[3]{a^6 b}$.

Therefore, *to reduce a mixed surd to an entire surd, raise the coefficient of the surd to the power indicated by the index of the radical, and introduce it as a factor under the radical sign.*

That is, $a\sqrt[n]{b} = \sqrt[n]{a^n b}$.

Reduce $3x\sqrt[3]{xy}$ and $-2a\sqrt{ab}$ to entire surds.

$$3x\sqrt[3]{xy} = \sqrt[3]{(3x)^3 xy} = \sqrt[3]{27x^4y}.$$

$$-2a\sqrt{ab} = -\sqrt{(2a)^2 ab} = -\sqrt{4a^3b}.$$

Exercise 107. Reduction of a Mixed Surd

Examples 1 to 3, oral — Examples 4 to 33, written

1. Reduce to entire surds: $2\sqrt{a}$; $a\sqrt{a}$; $ab\sqrt[3]{a^2b^2}$.
2. Reduce to entire surds: $x\sqrt[3]{y}$; $x\sqrt[3]{x}$; $(x+y)\sqrt[3]{(x+y)^2}$.
3. Reduce to entire surds: $3a\sqrt{2}$; $3x\sqrt{2xy}$; $5x^2\sqrt{x}$.

Reduce the following mixed surds to entire surds:

4. $2a^2\sqrt{7abc}$.
5. $13p^3\sqrt{2pqr}$.
6. $17mn\sqrt{9mn}$.
7. $7a^2b^2\sqrt[3]{3a^2b^2}$.
8. $5x^2y^3\sqrt[3]{5xy^2}$.
9. $7m^3n\sqrt[4]{m^3n}$.
10. $-11a\sqrt[3]{21a^2}$.
11. $-2x\sqrt[4]{x^2y^3}$.
12. $11pq\sqrt[4]{2p^2q^3}$.
13. $3abc\sqrt[5]{ab^2c^3}$.
14. $-\frac{2}{3}a\sqrt[3]{27a^2b^2}$.
15. $10ab\sqrt[5]{2a^4b^4}$.
16. $2ab\sqrt[7]{ab}$.
17. $3xy\sqrt[4]{x^3y^3}$.
18. $8m\sqrt[3]{4mn^2}$.
19. $2pq\sqrt[3]{p^2q^3}$.
20. $m^2n^2\sqrt[9]{mn}$.
21. $x^2y^3\sqrt[10]{x^3y^2}$.
22. $12ax\sqrt[4]{3a^3x}$.
23. $11mn\sqrt[4]{m^3n^3}$.
24. $a^2b^2c^2\sqrt[5]{a^2b^2c^2}$.
25. $-b^4c^4\sqrt{a^3b^3}$.
26. $-p^2q\sqrt[3]{p^2q}$.
27. $-2axy\sqrt[5]{2axy}$.
28. $3pqr\sqrt[7]{pqr}$.
29. $(a+b)\sqrt{a+b}$.
30. $(a-b)\sqrt{a-b}$.
31. $(x^2+y^2)\sqrt{x}$.
32. $x\sqrt{x^2+y^2}$.
33. $(x+y)\sqrt{x-y}$.

196. The *n*th Root of a Quotient. If we square $\frac{\sqrt{2}}{\sqrt{3}}$ and $\sqrt{\frac{2}{3}}$ we obtain the same result, thus:

$$\left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2 = \frac{\sqrt{2}\cdot\sqrt{2}}{\sqrt{3}\cdot\sqrt{3}} = \frac{2}{3}, \quad \text{and} \quad \left(\sqrt{\frac{2}{3}}\right)^2 = \frac{2}{3}. \qquad \S\,187$$

We therefore see that $\sqrt{\frac{2}{3}}$ equals $\frac{\sqrt{2}}{\sqrt{3}}$. In like manner,

The nth root of the quotient of two numbers equals the quotient of their nth roots.

Stated in symbols, $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$.

For example, $\sqrt[3]{\frac{27a^3b^6c}{125x^6y^9}} = \frac{\sqrt[3]{27a^3b^6c}}{\sqrt[3]{125x^6y^9}} = \frac{3ab^2\sqrt[3]{c}}{5x^2y^3}$.

Exercise 108. Roots of Quotients

Examples 1 and 2, oral — Examples 3 to 14, written

1. Simplify $\sqrt{\frac{4}{9}}$; $\sqrt{\frac{25}{36}}$; $\sqrt{\frac{25a^4}{49x^8y^{10}}}$; $\sqrt{\frac{36a^6b^8}{121x^4y^{10}}}$; $\sqrt[6]{\frac{a^{12}b}{x^{24}y^6}}$

2. Simplify $\sqrt[3]{\frac{8}{27}}$; $\sqrt[3]{\frac{27a^3}{125b^3}}$; $\sqrt[4]{\frac{1}{16}}$; $\sqrt[5]{\frac{a^{10}b^{15}}{x^{15}y^{20}}}$; $\sqrt[3]{-\frac{125a^6b^9}{x^9y^{12}z^{15}}}$

Simplify the following surds:

3. $5\sqrt{\frac{9a^2b}{16x^2y^4}}$.

4. $7\sqrt[3]{\frac{8a^6b}{27x^6y^{12}}}$.

5. $8ab\sqrt[3]{-\frac{a^3b^3}{x^6y^6}}$.

6. $4p^2\sqrt[3]{-\frac{p^9q}{x^{12}y^{15}}}$.

7. $5a^2\sqrt[3]{\frac{a^3b^3}{c^3d^3}}$.

8. $2xy\sqrt[3]{\frac{1}{8x^3y^3}}$.

9. $3ab\sqrt[3]{\frac{x^2y^2}{27a^6b^6}}$.

10. $2m\sqrt[4]{\frac{4a^2b^2}{16m^4}}$.

11. $\sqrt{\frac{(a+b)^2}{(a-b)^2}}$.

12. $\sqrt{\frac{(a-b)^4}{(a+b)^6}}$.

13. $\sqrt[6]{\frac{(x+2y)^2}{(y+2x)^6}}$.

14. $\sqrt[5]{\frac{(a+b)^6}{243(a-b)^5}}$.

197. The *m*th Root of the *n*th Root. If we separate a number into two equal factors, and each of these factors into three equal factors, we thereby separate the number into six equal factors.

For example, $64 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$. If we separate 64 into two equal factors, each factor equals $2 \cdot 2 \cdot 2$, or 8; and if we separate each of these into three equal factors, we have 2 for each factor, and this is one of the six equal factors of 64.

That is, $\sqrt[3]{\sqrt{64}} = \sqrt[3]{8} = 2$, and hence $\sqrt[3]{\sqrt{64}} = \sqrt[6]{64}$.

Therefore the cube root of the square root is the sixth root.

In the same way, *the mth root of the nth root of a number equals the mnth root of the number.*

That is, $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$.

Thus $\sqrt{\sqrt[3]{729}} = \sqrt[6]{729}$, and $\sqrt[3]{\sqrt{729}} = \sqrt[6]{729}$.

Similarly, $\sqrt[6]{49} = \sqrt[3]{\sqrt{49}} = \sqrt[3]{7}$; $\sqrt[10]{32} = \sqrt{\sqrt[5]{32}} = \sqrt{2}$.

Similarly, $\sqrt[mn]{a^{nx}} = \sqrt[m]{\sqrt[n]{a^{nx}}} = \sqrt[m]{a^x}$.

Exercise 109. Radicals

Examples 1 to 3, oral — Examples 4 to 15, written

1. Express as a square root: $\sqrt[4]{4}$; $\sqrt[6]{125}$; $\sqrt[6]{8}$; $\sqrt[8]{81}$; $\sqrt[10]{243}$.
2. Express as a cube root: $\sqrt[6]{9}$; $\sqrt[6]{25}$; $\sqrt[9]{8}$; $\sqrt[9]{27}$; $\sqrt[15]{32}$.
3. Express as a single root: $\sqrt{\sqrt{2}}$; $\sqrt{\sqrt[3]{7}}$; $\sqrt[3]{\sqrt{7}}$; $\sqrt[5]{\sqrt[6]{3}}$.

Express as a single root:

4. $\sqrt{\sqrt[3]{a^2+b^2}}$.
5. $\sqrt[3]{\sqrt{x^2-y^2}}$.
6. $\sqrt[n]{\sqrt[2m]{x}}$.
7. $\sqrt[n+7]{\sqrt[2n-3]{3x}}$.
8. $\sqrt[2a-7]{\sqrt[3a+2]{x}}$.
9. $\sqrt[2n-1]{\sqrt[3n+4]{a}}$.

Express as a simpler root:

10. $\sqrt[6]{121}$.
11. $\sqrt[6]{1331}$.
12. $\sqrt[8]{a^4b^6c^8}$.
13. $\sqrt[10]{32a^{10}b^{15}c^{20}d^{25}}$.
14. $\sqrt[9]{a^3b^6c^9}$.
15. $\sqrt[8]{(a+b)^2(a-b)^2}$.

198. Fractional Exponent. The positive integral exponent shows how many times the number which it affects is taken as a factor.

Thus $x^2 = xx$, the exponent showing that x is taken twice as a factor.

There are also fractional exponents, but these must have a different meaning from integral exponents, for we cannot take x half of a time as a factor any more than we can pick up a book half of a time.

Since we have certain laws of positive integral exponents (§ 167), it would be illogical and inconvenient if we should give to the fractional exponent a meaning such that these laws should not also apply to them. Therefore

Such a meaning must be given to fractional exponents as will make the laws of exponents valid for them as well as for positive integral exponents.

Since $a^m \cdot a^m = a^{m+m}$,

we must have $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$.

But $\sqrt{a} \cdot \sqrt{a} = a$.

We therefore define $a^{\frac{1}{2}}$ to mean $\sqrt{a}$.

And since $(a^m)^n = a^{mn}$,

we must have $(a^{\frac{1}{2}})^3 = a^{\frac{3}{2}}$.

That is, we define $a^{\frac{3}{2}}$ to mean $(\sqrt{a})^3$.

Therefore we see that the following meaning must be given to a fractional exponent if the laws of exponents are to remain valid.

In a fractional exponent the numerator indicates the power of the number affected, and the denominator indicates the root.

Thus $4^{\frac{1}{2}}$ means the same as $\sqrt{4}$ and equals 2;

$8^{\frac{2}{3}}$ means the same as $(\sqrt[3]{8})^2$, or as $\sqrt[3]{8^2}$ and equals 4,

and $a^{\frac{m}{n}}$ means the same as $(\sqrt[n]{a})^m$, or as $\sqrt[n]{a^m}$.

Sometimes it is much easier to use fractional exponents than radicals. In simple cases, like $\sqrt{2}$ or $\sqrt[3]{8}$, it is easier to write the radical sign.

Exercise 110. Fractional Exponents

Examples 1 to 5, oral — Examples 6 to 56, written

1. What is the meaning of $a^{\frac{1}{2}}$? $a^{\frac{1}{3}}$? $a^{\frac{1}{5}}$? $a^{\frac{1}{n}}$?
2. What is the meaning of $r^{\frac{2}{3}}$? $(abc)^{\frac{2}{3}}$? $a^{\frac{1}{2}}b^{\frac{2}{3}}$? $a^{\frac{3}{4}}b^{\frac{4}{5}}c^{\frac{1}{2}}$?
3. What is the value of $4^{\frac{1}{2}}$? $36^{\frac{1}{2}}$? $81^{\frac{1}{2}}$? $144^{\frac{1}{2}}$? $400^{\frac{1}{2}}$?
4. What is the value of $16^{\frac{3}{4}}$? $32^{\frac{1}{5}}$? $32^{\frac{2}{5}}$? $32^{\frac{4}{5}}$? $125^{\frac{2}{3}}$?
5. What is the value of $8^{\frac{1}{3}} \cdot 32^{\frac{1}{5}}$? $8^{\frac{2}{3}} \cdot 27^{\frac{1}{3}}$? $8^{\frac{2}{3}} \cdot 27^{\frac{2}{3}}$?

Express with radical signs:

6. $a^{\frac{5}{6}}$.
7. $x^{\frac{7}{2}}$.
8. $m^{\frac{3}{5}}$.
9. $2\,a^{\frac{1}{2}}$.
10. $(2\,a)^{\frac{1}{2}}$.
11. $2^{\frac{1}{2}}a^{\frac{1}{2}}$.
12. $7x^{\frac{3}{5}}$.
13. $(7x)^{\frac{2}{3}}$.
14. $7^{\frac{3}{5}}x^{\frac{2}{3}}$.
15. $a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{1}{4}}$.
16. $p^{\frac{1}{2}}q^{\frac{2}{3}}r^{\frac{3}{4}}$.
17. $x^{\frac{1}{a}}y^{\frac{1}{b}}z^{\frac{1}{c}}$.

Simplify the following:

18. $81^{\frac{1}{2}}$.
19. $81^{\frac{1}{4}}$.
20. $81^{\frac{3}{4}}$.
21. $125^{\frac{1}{3}}$.
22. $125^{\frac{2}{3}}$.
23. $125^{\frac{4}{3}}$.
24. $-343^{\frac{1}{3}}$.
25. $-343^{\frac{2}{3}}$.
26. $(-32\,a)^{\frac{1}{5}}$.
27. $(-32\,a)^{\frac{2}{5}}$.
28. $(-32\,a)^{\frac{3}{5}}$.
29. $(-32\,a)^{\frac{4}{5}}$.
30. $32^{\frac{1}{5}} \cdot 27^{\frac{2}{3}}$.
31. $(\frac{1}{25})^{\frac{1}{2}} \cdot (\frac{1}{49})^{\frac{1}{2}}$.
32. $4^{\frac{1}{2}} \cdot (\frac{1}{4})^{\frac{1}{2}} \cdot 9^{\frac{1}{2}}$.
33. $125^{\frac{2}{3}} \cdot 625^{\frac{1}{4}}$.
34. $121^{\frac{1}{2}} \cdot 144^{\frac{1}{2}}$.
35. $(\frac{1}{64})^{\frac{1}{2}} \cdot (\frac{1}{1331})^{\frac{2}{3}}$.

Express with fractional exponents:

36. $\sqrt{a}$.
37. $\sqrt[3]{a^2}$.
38. $\sqrt[3]{a^2b}$.
39. $\sqrt[3]{2\,ab^2}$.
40. $\sqrt[4]{a^3}$.
41. $\sqrt[4]{a^3b}$.
42. $\sqrt[4]{3\,a^3b^5}$.
43. $\sqrt[5]{a^2b^2}$.
44. $\sqrt[5]{a^2b^3c^4}$.
45. $\sqrt[5]{x^4yz^6}$.
46. $2\sqrt{a}\sqrt[3]{b}$.
47. $3\sqrt[3]{m}\sqrt[4]{n^3}$.
48. $5^2 \cdot \sqrt[3]{m^2}\sqrt[4]{n^5}$.
49. $2\,a^2\sqrt[4]{p}\sqrt[5]{q^2}$.
50. $\sqrt[6]{a}\sqrt[7]{b}$.
51. $a\sqrt[3]{(mn)^2}$.
52. $\sqrt{(a-b)^3}$.
53. $a\sqrt{b} \cdot c^2\sqrt[3]{d}$.
54. $\sqrt[n]{a} \cdot \sqrt[m]{b}$.
55. $\sqrt[n]{a^m} \cdot \sqrt[p]{b^q}$.
56. $\sqrt[n]{a^{2m}} \cdot \sqrt[p]{b^q c^r}$.

199. Similar Surds. Surds which, when reduced to their simplest forms, have the same surd factor are called *similar surds.*

For example, $3\sqrt{2}$, $4\sqrt{2}$, and $9\sqrt{2}$ are similar surds.

200. Addition and Subtraction of Surds. The principles of addition and subtraction of similar surds are the same as those of other algebraic expressions.

Thus $$3\sqrt{2} + 4\sqrt{2} = 7\sqrt{2},$$

and $$a\sqrt[n]{x} + b\sqrt[n]{x} - c\sqrt[n]{x} = (a + b - c)\sqrt[n]{x}.$$

Simplify $\sqrt{243} - \sqrt{75}$.

$$\sqrt{243} = \sqrt{81 \cdot 3} = 9\sqrt{3};$$
$$\sqrt{75} = \sqrt{25 \cdot 3} = 5\sqrt{3}.$$
$$\sqrt{243} - \sqrt{75} = 9\sqrt{3} - 5\sqrt{3} = 4\sqrt{3}.$$

Exercise 111. Addition and Subtraction of Surds

Examples 1 to 3, oral — Examples 4 to 12, written

1. Simplify $3\sqrt{a} + 9\sqrt{a}$; $17\sqrt[3]{x^2} - 9\sqrt[3]{x^2}$.
2. Simplify $39\sqrt[3]{4} + 17\sqrt[3]{4}$; $41\sqrt[5]{3} - 16\sqrt[5]{3}$.
3. Simplify $41\sqrt[5]{-a^2b} + 17\sqrt[5]{-a^2b}$; $43\sqrt[7]{x^2y^2} - 19\sqrt[7]{x^2y^2}$.

Simplify the following:

4. $2\sqrt{x} + 3\sqrt{x} + 9\sqrt{x} - 4\sqrt{x} + 2\sqrt{x} - 6\sqrt{x}$.
5. $3\sqrt{a} - 7\sqrt{b} + 4\sqrt{a} - 2\sqrt{b} + 5\sqrt{a} - \sqrt{b}$.
6. $3\sqrt{4m} - 2\sqrt{9m} + 2\sqrt{25m} - 4\sqrt{49m} + 12\sqrt{81m}$.
7. $5\sqrt{9x} + 7\sqrt[3]{2x} - 3\sqrt{25x} + 4\sqrt[3]{16x} + 10\sqrt{100x}$.
8. $3\sqrt{8} + 2\sqrt{32} + 7\sqrt{50} - 6\sqrt{162} + 9\sqrt{98} + 7\sqrt{242}$.
9. $\sqrt{(a+b)^2c} + \sqrt{(a-b)^2c} + \sqrt{a^2c} - \sqrt{b^2c} + (a+b)c$.
10. $99^{\frac{1}{2}} + 44^{\frac{1}{2}} + 176^{\frac{1}{2}} + 275^{\frac{1}{2}} - 396^{\frac{1}{2}} + 2\sqrt{11}$.
11. $\sqrt[3]{16} + \sqrt[3]{54} + \sqrt[3]{2662} - \sqrt[3]{128} + \sqrt[3]{-250}$.
12. $\sqrt[4]{32} + 2\sqrt[4]{162} + 3\sqrt[4]{512} + 5\sqrt[4]{1250} - \frac{1}{6}\sqrt[4]{2592}$.

201. Reduction of Surds to a Common Index. For the present it will be assumed that a fractional exponent may be reduced like an ordinary fraction. This is discussed fully in Book II. This fact being assumed, we see that

$$\sqrt[5]{a^4} = a^{\frac{4}{5}} = a^{\frac{12}{15}} = \sqrt[15]{a^{12}},$$

and that

$$\sqrt[3]{a^2} = a^{\frac{2}{3}} = a^{\frac{10}{15}} = \sqrt[15]{a^{10}}.$$

In the same way we may reduce any two surds to surds having the same index.

That is,

$$\sqrt[m]{a^k} = a^{\frac{k}{m}} = a^{\frac{kn}{mn}} = \sqrt[mn]{a^{kn}}.$$

Reduce $\sqrt[3]{4}$ and $\sqrt[4]{27}$ to surds having a common index.

$$\sqrt[3]{4} = \sqrt[3]{2^2} = 2^{\frac{2}{3}};$$

$$\sqrt[4]{27} = \sqrt[4]{3^3} = 3^{\frac{3}{4}}.$$

Since the L.C.D. of $\frac{2}{3}$ and $\frac{3}{4}$ is 12, we reduce to twelfths.

Then

$$\sqrt[3]{4} = 2^{\frac{2}{3}} = 2^{\frac{8}{12}} = \sqrt[12]{2^8};$$

$$\sqrt[4]{27} = 3^{\frac{3}{4}} = 3^{\frac{9}{12}} = \sqrt[12]{3^9}.$$

Exercise 112. Reduction of Surds to a Common Index

Examples 1 to 3, oral — Examples 4 to 16, written

1. Express as a twelfth root: $a^{\frac{1}{2}}$; $b^{\frac{1}{3}}$; $c^{\frac{1}{4}}$; $d^{\frac{1}{6}}$; $\sqrt{x}$; $\sqrt[3]{y}$; $\sqrt[4]{z}$.
2. Express as a tenth root: $x^{\frac{1}{2}}$; $y^{\frac{1}{5}}$; $\sqrt{z}$; $\sqrt[5]{w}$; $\sqrt[5]{m^2}$; $\sqrt[5]{n^3}$.
3. Express $x^{\frac{1}{3}}$ and $y^{\frac{3}{4}}$ with a common root index.

Reduce to surds having a common root index:

4. $\sqrt{5}$, $\sqrt[3]{2}$.
5. $\sqrt[3]{2}$, $\sqrt[4]{3}$.
6. $\sqrt[4]{a^3}$, $\sqrt[5]{b^3}$.
7. $\sqrt[3]{p^2}$, $\sqrt[4]{q^3}$.
8. $a^{\frac{1}{2}}$, $b^{\frac{2}{3}}$.
9. $x^{\frac{1}{3}}$, $\sqrt{y}$.
10. $2a^{\frac{2}{3}}$, $\sqrt[4]{a}$.
11. $3x^{\frac{3}{4}}$, $\sqrt{x}$.
12. $2\sqrt{2}$, $3\sqrt[3]{2}$, $2\sqrt[4]{3}$.
13. $3\sqrt{2}$, $5\sqrt[3]{3}$, $3\sqrt[4]{7}$.
14. $2\sqrt{2}$, $3\sqrt[4]{3}$, $5\sqrt[6]{4}$.
15. $a^{\frac{2}{3}}$, $b^{\frac{1}{4}}$, $c^{\frac{5}{6}}$, $\sqrt{d}$.

16. Reduce to surds having a common index, and then arrange in order of magnitude: $\sqrt{8}$, $\sqrt[3]{10}$, $\sqrt[6]{81}$; $\sqrt{3}$, $\sqrt[3]{5}$, $\sqrt[6]{23}$; $\sqrt[3]{7}$, $\sqrt[4]{15}$, $\sqrt{2}$.

202. Multiplication of Surds. Since we have found (§ 188) that $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$, therefore

To multiply one surd by another, express the surds with a common index. Then find the product of the coefficients for the required coefficient, and the product of the surd factors for the required surd factor.

In all work with radicals the results should be reduced to their simplest forms.

Simplify $4\sqrt{3} \cdot 2\sqrt[3]{9}$.

$$4\sqrt{3} \cdot 2\sqrt[3]{9} = 4\sqrt[6]{3^3} \cdot 2\sqrt[6]{9^2} = 4\sqrt[6]{3^3} \cdot 2\sqrt[6]{3^4}$$
$$= 8\sqrt[6]{3^7} = 8 \cdot 3\sqrt[6]{3} = 24\sqrt[6]{3}.$$

We may also express the numbers with fractional exponents, thus:

$$4 \cdot 3^{\frac{1}{2}} \cdot 2 \cdot 9^{\frac{1}{3}} = 4 \cdot 2 \cdot 3^{\frac{1}{2}} \cdot 3^{\frac{2}{3}} = 8 \cdot 3^{\frac{1}{2}+\frac{2}{3}}$$
$$= 8 \cdot 3^{\frac{7}{6}} = 8 \cdot 3 \cdot 3^{\frac{1}{6}} = 24 \cdot 3^{\frac{1}{6}}.$$

Exercise 113. Multiplication of Radicals

Examples 1 to 3, oral — Examples 4 to 20, written

1. Simplify: $a^{\frac{1}{2}}a^{\frac{1}{2}}$; $a^{\frac{1}{2}}a^{\frac{2}{3}}$; $a^{\frac{1}{2}}a^{\frac{1}{3}}$; $\sqrt{x} \cdot \sqrt[3]{x}$.
2. Simplify: $a^{\frac{1}{2}}a^{\frac{2}{3}}$; $\sqrt[5]{x} \cdot \sqrt[3]{x^2}$; $a^{\frac{1}{3}}a^{\frac{1}{6}}$; $\sqrt[3]{a} \cdot \sqrt[6]{a^5}$.
3. Simplify: $\sqrt{3} \cdot \sqrt{12}$; $\sqrt{2} \cdot \sqrt{18}$; $\sqrt{5} \cdot \sqrt{20}$; $\sqrt{3} \cdot \sqrt{27}$.

4. $\sqrt{20} \cdot \sqrt{30}$.
5. $\sqrt[3]{5} \cdot \sqrt[3]{135}$.
6. $\sqrt[3]{a} \cdot \sqrt[5]{a^2}$.
7. $2\sqrt{7} \cdot 7\sqrt{2}$.
8. $\sqrt{18} \cdot \sqrt[3]{6}$.
9. $2\sqrt{3} \cdot 3\sqrt[3]{2}$.
10. $4\sqrt{5} \cdot 2\sqrt[3]{3}$.
11. $\sqrt[3]{9} \cdot 27\sqrt{3}$.
12. $\sqrt[3]{16} \cdot 5\sqrt[3]{4}$.
13. $\sqrt[3]{-16} \cdot \sqrt{18}$.
14. $\sqrt{6} \cdot \sqrt{864}$.
15. $\sqrt[3]{9} \cdot \sqrt{800}$.
16. $(\sqrt{2} + 3\sqrt{18} + 9\sqrt{50} - 7\sqrt{32})\sqrt{2}$.
17. $(\sqrt[3]{24} + 8\sqrt[3]{81} - 2\sqrt[3]{375} - 3\sqrt[3]{192})\sqrt[3]{3}$.
18. $(\sqrt{12} + 2\sqrt{27} + 8\sqrt{75} - 5\sqrt{108} + 4\sqrt{147} + \sqrt{192})\sqrt{3}$
19. $(\sqrt{5} + \sqrt{20} + 9\sqrt{45} - 2\sqrt{80} - 3\sqrt{125} + 5\sqrt{180})\sqrt{5}$.
20. $(\sqrt[3]{2} + 5\sqrt[3]{54} - 6\sqrt[3]{128} + 5\sqrt[3]{16} - 2\sqrt[3]{250} + 7\sqrt[3]{432})\sqrt[3]{2}$.

203. Multiplication of Polynomial Radicals. Polynomial radicals are multiplied like ordinary polynomials. For example,

$$\begin{array}{r} 3\sqrt{5}+\ 2\sqrt{3} \\ 7\sqrt{5}-\ 4\sqrt{3} \\ \hline 105+14\sqrt{15} \\ \quad -12\sqrt{15}-24 \\ \hline 105+\ 2\sqrt{15}-24 \end{array} = 81+2\sqrt{15}.$$

Exercise 114. Multiplication of Polynomial Radicals

Examples 1 to 4, oral — Examples 5 to 24, written

1. Multiply $2\sqrt{3}+3\sqrt{2}$ by 2; by 3; by 5; by -4.
2. Multiply $\sqrt{2}+1$ by $\sqrt{2}-1$; by $\sqrt{2}+1$.
3. Multiply $\sqrt{2}+\sqrt{3}$ by $\sqrt{2}-\sqrt{3}$; by $\sqrt{2}+\sqrt{3}$.
4. Square $\sqrt{5}+\sqrt{2}$; $\sqrt{5}-\sqrt{2}$; $\sqrt{2}+\sqrt{5}$; $\sqrt{2}-\sqrt{5}$.

5. $(\sqrt{8}+\sqrt{5})(\sqrt{8}-\sqrt{5})$.
6. $(\sqrt{8}+\sqrt{7})(\sqrt{8}+2\sqrt{7})$.
7. $(\sqrt{10}-\sqrt{3})(\sqrt{10}+4\sqrt{3})$.
8. $(7-3\sqrt{6})(9+7\sqrt{6})$.
9. $(a+2\sqrt{b})(3a-4\sqrt{b})$.
10. $(x-3\sqrt{y})(2x+7\sqrt{y})$.
11. $\sqrt{7+\sqrt{18}}\cdot\sqrt{7-\sqrt{18}}$.
12. $\sqrt{7+2\sqrt{6}}\cdot\sqrt{7-2\sqrt{6}}$.
13. $\sqrt{9+\sqrt{17}}\cdot\sqrt{9-\sqrt{17}}$.
14. $(3+\sqrt[3]{2})(7-2\sqrt[3]{2})$.
15. $(4+\sqrt[4]{3})(5-2\sqrt[4]{3})$.
16. $(\sqrt[5]{2}+\sqrt[5]{3})(\sqrt[5]{2}-\sqrt[5]{3})$.
17. $(12\sqrt{5}-11\sqrt{3})(13\sqrt{5}+19\sqrt{3})$.
18. $(5+9\sqrt{3}+6\sqrt{5})(8-7\sqrt{3}-9\sqrt{5})$.
19. $(2+7\sqrt{6}-12\sqrt{5})(7-3\sqrt{6}+8\sqrt{5})$.
20. $(2\sqrt{7}-8\sqrt{28}+5\sqrt{63})(2\sqrt{7}-8\sqrt{28}-5\sqrt{63})$.
21. $(\sqrt[3]{9}-7\sqrt[3]{72}+6\sqrt[3]{1125})(3\sqrt[3]{2}+8\sqrt[3]{3}-4\sqrt[3]{9})$.
22. $(\sqrt{a}+\sqrt{b})(a^2+ab+b^2)(\sqrt{a}-\sqrt{b})$.
23. $(\sqrt{x}-\sqrt{y})(x+y)(\sqrt{x}+\sqrt{y})(x^2+y^2)$.
24. $(a-\sqrt{ab}+b)(a+\sqrt{ab}+b)(a^2-ab+b^2)$.

204. Division of Radicals. If, in the solution of a problem, we find $\frac{3}{\sqrt{5}}$ as the result, we have an awkward numerical expression. If we extract the square root of 5 to any number of decimal places, we have a long divisor. But if we multiply both terms by $\sqrt{5}$, we have $\frac{3\sqrt{5}}{5}$, and the division is much simpler, as stated in § 182.

Comparing the two operations we have

$$(1)\quad \frac{3}{\sqrt{5}} = \frac{3}{2.236} = 1.3416, \text{ nearly.}$$

$$(2)\quad \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5} = \frac{6.708}{5} = 1.3416, \text{ nearly.}$$

The work in (1) will be found more difficult than that in (2).

Therefore, *if the denominator of a fraction contains a surd, multiply both terms of the fraction by a number that will make the denominator rational.*

This is called *rationalizing the denominator*, the expression meaning that the denominator is to be made rational without changing the value of the fraction.

1. Divide 3 by $\sqrt{2}$.

$\frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$. The result should be left in this form unless the extraction of the root is required.

2. Divide $\sqrt{2}$ by $\sqrt{3}$.

$$\frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{2}\cdot\sqrt{3}}{3} = \frac{1}{3}\sqrt{6}.$$

3. Simplify $\sqrt[3]{\frac{3}{4}}$.

$$\sqrt[3]{\tfrac{3}{4}} = \sqrt[3]{\tfrac{6}{8}} = \tfrac{1}{2}\sqrt[3]{6}.$$

4. Divide $7\sqrt[3]{50}$ by $\sqrt{2}$.

$$\frac{7\sqrt[3]{50}}{\sqrt{2}} = \frac{7\sqrt{2}\sqrt[3]{50}}{2} = \frac{7\sqrt[6]{8\cdot 2500}}{2} = \frac{7}{2}\sqrt[6]{20{,}000}.$$

Exercise 115. Division of Radicals

Examples 1 to 4, oral — Examples 5 to 28, written

1. Simplify $\frac{1}{\sqrt{2}}$; $\frac{2}{\sqrt{2}}$; $\frac{1}{\sqrt[3]{2^2}}$; $\frac{1}{\sqrt[3]{3^2}}$; $\frac{2}{\sqrt{5}}$; $\frac{3}{\sqrt{7}}$.

2. Simplify $\sqrt{\frac{2}{3}}$; $\sqrt{\frac{3}{5}}$; $\sqrt[3]{\frac{3}{4}}$; $\sqrt[3]{\frac{5}{9}}$; $\sqrt[3]{\frac{3}{5^2}}$; $\sqrt[3]{\frac{7}{25}}$.

3. Simplify $\sqrt[4]{\frac{2}{3^3}}$; $\sqrt[4]{\frac{5}{27}}$; $\sqrt[5]{\frac{3}{2^4}}$; $\sqrt[5]{\frac{5}{16}}$; $\sqrt[6]{\frac{2}{3^5}}$.

4. Simplify $\sqrt{a^3} \div \sqrt{a}$; $\sqrt{a^5} \div \sqrt{a^3}$; $\sqrt{ab} \div \sqrt{a}$.

Divide:

5. $\sqrt[3]{63}$ by $\sqrt[3]{3^2}$.
6. $\sqrt[3]{3^2}$ by $\sqrt[3]{2^2}$.
7. $75^{\frac{1}{3}}$ by $5^{\frac{1}{3}}$.
8. $25^{\frac{3}{5}}$ by $5^{\frac{3}{5}}$.
9. $\sqrt{\frac{2}{3}}$ by $\sqrt{\frac{3}{5}}$.
10. $\sqrt[3]{1\frac{1}{2}}$ by $\sqrt[3]{1\frac{3}{8}}$.
11. 7 by $\sqrt{21}$.
12. 9 by $\sqrt{21}$.
13. $\sqrt{a}$ by $\sqrt[3]{a}$.
14. $\sqrt[3]{a^2}$ by $\sqrt{a}$.
15. ab^4 by $\sqrt{ab}$.
16. xy by $\sqrt[3]{x^2y^2}$.

17. From $a\sqrt{x} - b\sqrt{x} + c\sqrt{x}$ remove the factor $\sqrt{x}$. What is the quotient of $a\sqrt{x} - b\sqrt{x} + c\sqrt{x}$ divided by $\sqrt{x}$?

Divide:

18. $8\sqrt{50} - 7\sqrt{32} + 2\sqrt{18} - 6\sqrt{98}$ by $2\sqrt{2}$.
19. $9\sqrt[3]{24} + 6\sqrt[3]{81} - 3\sqrt[3]{192} + 12\sqrt[3]{375}$ by $3\sqrt[3]{3}$.
20. $56\sqrt{30} - 84\sqrt{10} + 100\sqrt{15}$ by $4\sqrt{35}$.
21. $10\sqrt{14} - 5\sqrt{63} + 4\sqrt{28} + 6\sqrt{7}$ by $2\sqrt{7}$.
22. $6\sqrt{33} + 5\sqrt{22} - 7\sqrt{66} + 8\sqrt{44}$ by $\sqrt{11}$.
23. $a\sqrt[3]{a^2b^5} - b\sqrt[3]{a^5b^2} + c\sqrt[3]{a^5b^5} - d\sqrt[3]{a^8b^2}$ by $\sqrt[3]{a^2b^2}$.
24. $a^{\frac{1}{2}}b^{\frac{3}{2}} + a^{\frac{1}{2}}b^{\frac{1}{2}} - a^{\frac{3}{2}}b^{\frac{1}{2}} + a^{\frac{1}{2}}b^{\frac{1}{2}}$ by $a^{\frac{1}{2}}b^{\frac{1}{2}}$.
25. $4\sqrt{15} + 16\sqrt{35} - 10\sqrt{30} + 12\sqrt{50}$ by $2\sqrt{5}$.
26. $9\sqrt[3]{35} + 15\sqrt[3]{45} + 30\sqrt[3]{55} - 21\sqrt[3]{60}$ by $3\sqrt[3]{5}$.
27. $abc\sqrt[3]{abc} + a^2bc\sqrt[3]{a^2bc} - ab^2c\sqrt[3]{ab^2c} + \sqrt[3]{abc^2}$ by $\sqrt[3]{abc}$.
28. $(a+b)\sqrt{a+b} - (a-b)\sqrt{a^2-b^2} + ab\sqrt{a^2b+ab^2}$ by $\sqrt{a+b}$.

205. Division by a Binomial Surd. If the divisor or the denominator of a fraction is a binomial involving only quadratic surds, it may be rationalized by the process shown in the following example:

Simplify $\dfrac{2+3\sqrt{2}}{3-5\sqrt{2}}$.

Multiply both terms of the fraction by $3+5\sqrt{2}$, because the denominator will then become the difference of two squares. Then

$$\frac{2+3\sqrt{2}}{3-5\sqrt{2}}=\frac{(3+5\sqrt{2})(2+3\sqrt{2})}{(3+5\sqrt{2})(3-5\sqrt{2})}=\frac{36+19\sqrt{2}}{9-50}=-\frac{1}{41}(36+19\sqrt{2}).$$

The binomial $3+5\sqrt{2}$ is called the *conjugate* of $3-5\sqrt{2}$. In such cases we multiply both terms of the fraction by the conjugate of the denominator, thus rationalizing the denominator.

Exercise 116. Division by a Binomial Surd

Examples 1 to 3, oral — Examples 4 to 36, written

1. What is the conjugate of $3-\sqrt{7}$? of $4+\sqrt{5}$?

2. By what should $\sqrt{7}+\sqrt{5}$ be multiplied that the product may be rational? What is the product?

3. Multiply $12+\sqrt{10}$ by $12-\sqrt{10}$; $\sqrt{51}+\sqrt{11}$ by $\sqrt{51}-\sqrt{11}$; $9\sqrt{2}-\sqrt{10}$ by $9\sqrt{2}+\sqrt{10}$.

Divide:

4. $\sqrt{3}$ by $\sqrt{5}-\sqrt{2}$.
5. 5 by $\sqrt{3}-\sqrt{2}$.
6. $\sqrt{2}$ by $\sqrt{5}-\sqrt{2}$.
7. $3+\sqrt{2}$ by $3-\sqrt{2}$.
8. $5+\sqrt{7}$ by $\sqrt{7}-5$.
9. $\sqrt{3}+\sqrt{2}$ by $\sqrt{3}-\sqrt{2}$.
10. $8+7\sqrt{5}$ by $2-3\sqrt{5}$.
11. $3+\sqrt{6}$ by $\sqrt{2}+\sqrt{3}$.
12. $5\sqrt{3}-3\sqrt{5}$ by $\sqrt{5}-\sqrt{3}$.
13. $7\sqrt{5}+5\sqrt{7}$ by $\sqrt{5}+\sqrt{7}$.
14. $2\sqrt{3}+\sqrt{6}$ by $\sqrt{3}+\sqrt{6}$.
15. $a+b\sqrt{x}$ by $a-b\sqrt{x}$.

16. Simplify $\dfrac{3+\sqrt{5}}{3-\sqrt{5}}$ and find the approximate value to two decimal places.

Rationalize the denominators:

17. $\dfrac{2+\sqrt{3}}{2-\sqrt{3}}$.

18. $\dfrac{2}{7-3\sqrt{5}}$.

19. $\dfrac{4-2\sqrt{2}}{4+2\sqrt{2}}$.

20. $\dfrac{5-3\sqrt{3}}{5+3\sqrt{3}}$.

21. $\dfrac{2+\sqrt{2}}{3-\sqrt{2}}$.

22. $\dfrac{4+\sqrt{7}}{5-\sqrt{7}}$.

23. $\dfrac{5-\sqrt{2}}{5+\sqrt{3}}$.

24. $\dfrac{2-\sqrt{5}}{3+\sqrt{2}}$.

25. $\dfrac{a+\sqrt{b}}{a-\sqrt{b}}$.

26. $\dfrac{a-\sqrt{b}}{a+\sqrt{b}}$.

27. $\dfrac{a+\sqrt{b}}{c-\sqrt{d}}$.

28. $\dfrac{x+y\sqrt{y}}{x-y\sqrt{y}}$.

29. Simplify $\dfrac{3+\sqrt{7}}{3-\sqrt{7}}$, and find the approximate value to three decimal places.

30. Simplify $\dfrac{5+2\sqrt{3}}{5-2\sqrt{3}}$, and find the approximate value to three decimal places.

31. Arrange in order of magnitude, beginning with the largest: $\dfrac{3+\sqrt{7}}{3-\sqrt{7}}$, $\dfrac{3+\sqrt{5}}{3-\sqrt{5}}$, and $\dfrac{4+\sqrt{6}}{4-\sqrt{6}}$.

32. In the fraction $\dfrac{2+\sqrt{3}+\sqrt{5}}{2+\sqrt{3}-\sqrt{5}}$, multiply both terms by $2+\sqrt{3}+\sqrt{5}$. Then rationalize the denominator of the resulting fraction.

33. Rationalize the denominator of the fraction $\dfrac{3+\sqrt{2}+\sqrt{3}}{3-\sqrt{2}+\sqrt{3}}$ by a method similar to that of Ex. 32.

34. Rationalize the denominator of the fraction $\dfrac{5+\sqrt{2}-\sqrt{3}}{5+\sqrt{2}+\sqrt{3}}$ by a method similar to that of Ex. 32.

35. Rationalize the denominator of the fraction $\dfrac{7+\sqrt{3}-\sqrt{2}}{7+\sqrt{3}+\sqrt{2}}$.

36. Rationalize the denominator and then find, to three decimal places, the approximate value of $\dfrac{3.5\sqrt{2}+\frac{1}{2}\sqrt{6}-1}{3.5\sqrt{2}+\frac{1}{2}\sqrt{6}+1}$.

206. Powers of Radicals. The powers of radicals are conveniently found by the following methods:

1. Find the square of $2\sqrt[3]{5^2}$ and the cube of $5\sqrt{x}$.

$$\left(2\sqrt[3]{5^2}\right)^2 = (2 \cdot 5^{\frac{2}{3}})^2 = 2^2 \cdot 5^{\frac{4}{3}} = 4 \cdot 5 \cdot 5^{\frac{1}{3}} = 20\sqrt[3]{5}.$$

$$\left(5\sqrt{x}\right)^3 = (5\,x^{\frac{1}{2}})^3 = 125\,x^{\frac{3}{2}} = 125\,xx^{\frac{1}{2}} = 125\,x\sqrt{x}.$$

2. Find the square of $2 + \sqrt{3}$.

$$\left(2 + \sqrt{3}\right)^2 = 4 + 4\sqrt{3} + 3$$
$$= 7 + 4\sqrt{3}.$$

Exercise 117. Powers of Radicals

Examples 1 to 4, oral — Examples 5 to 30, written

1. Raise to the second power: $\sqrt{7}$; $\sqrt[4]{3}$; $\sqrt{ab}$; $\sqrt[6]{a}$.
2. Raise to the third power: $\sqrt[3]{x}$; $\sqrt[3]{x^2}$; $x^{\frac{1}{3}}$; $x^{\frac{5}{3}}$.
3. Raise to the fourth power: $\sqrt{a}$; $a^{\frac{1}{4}}$; $a^{\frac{3}{4}}$; $\sqrt[4]{m^3}$.
4. Raise to the fifth power: $\sqrt[5]{2x}$; $x^{\frac{3}{5}}$; $\sqrt[5]{m^3}$; $\sqrt[5]{x^4y^4}$.

Perform the operations indicated:

5. $\left(\sqrt[3]{a^2}\right)^2$.
6. $\left(\sqrt[3]{a^2}\right)^4$.
7. $\left(\sqrt[4]{x^3}\right)^5$.
8. $\left(\sqrt[4]{x^2y^2}\right)^5$.
9. $\left(\sqrt[5]{a^2b^3c^4}\right)^6$.
10. $\left(\sqrt[3]{-a^2b^2}\right)^4$.
11. $\left(-\sqrt[3]{a^2b^2}\right)^4$.
12. $\left(-\sqrt[4]{111}\right)^8$.
13. $\left(\sqrt[6]{x}\right)^3$.
14. $\left(\sqrt[5]{x}\right)^4$.
15. $\left(\sqrt[6]{x}\right)^3$.
16. $\sqrt[3]{\sqrt{a}}$.
17. $\sqrt[3]{\sqrt[4]{a^3}}$.
18. $\sqrt[5]{\sqrt[3]{x^5}}$.
19. $\sqrt[4]{\sqrt{x^3}}$.
20. $\sqrt[3]{\sqrt[4]{a^{12}}}$.
21. $\left[(a+b)^{\frac{1}{2}}\right]^4$.
22. $\left[\sqrt{(a+b)^3}\right]^3$.
23. $(a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{1}{4}})^{12}$.
24. $(a^{\frac{1}{2}}b^{\frac{3}{4}}c^{\frac{3}{4}})^{12}$.
25. $\left[(1+x^2)^{\frac{1}{2}}\right]^6$.
26. $\left[(\sqrt{a-b})^{\frac{1}{2}}\right]^4$.
27. $\left[(\sqrt[3]{a+b})^{\frac{1}{2}}\right]^6$.
28. $\left[(\sqrt[5]{x+y})^{\frac{1}{2}}\right]^{30}$.
29. Multiply $\left(\sqrt[6]{4}\right)^2$ by $\sqrt[3]{4}$; by $\sqrt[3]{4^2}$; by $\left(\sqrt[3]{4}\right)^4$; by $\sqrt{4^3}$.
30. Divide $\left(\sqrt[4]{a^3b^3}\right)^6$ by $\sqrt[4]{a^3b^3}$; by $\left(\sqrt[4]{a^3b^3}\right)^3$; by $\left(\sqrt[6]{a^5b^5}\right)^{12}$.

207. Equations involving Radicals. Certain forms of equations containing radicals can be solved by the operations already studied. The following are types:

1. Solve the equation $x = 3 + \sqrt{x^2 - 45}$.

Subtracting 3, $x - 3 = \sqrt{x^2 - 45}$.

Squaring, $x^2 - 6x + 9 = x^2 - 45$.

Subtracting $x^2 + 9$, $-6x = -54$.

Dividing by -6, $x = 9$.

2. Solve the equation $\sqrt{x - 0.1} = \sqrt{x - 3.1} + 1$.

Squaring, $x - 0.1 = x - 3.1 + 2\sqrt{x - 3.1} + 1$.

Subtracting $x - 3.1 + 1$, $2 = 2\sqrt{x - 3.1}$.

Dividing by 2, $1 = \sqrt{x - 3.1}$.

Squaring, $1 = x - 3.1$.

Adding 3.1, $4.1 = x$.

Exercise 118. Equations involving Radicals

Examples 1 to 7, oral — Examples 8 to 19, written

1. Solve $\sqrt{x + 7} = 5$; $\sqrt{x - 7} = 3$; $\sqrt{2x - 5} = 5$.
2. Solve $\sqrt{x - 3} - 4 = 0$; $\sqrt{x - 9} - 5 = 0$; $\sqrt{x} - 2 = 7$.
3. Solve $\sqrt{x}(\sqrt{2} + 1) = \sqrt{2} + 1$; $\sqrt{2x} + \sqrt{x} = \sqrt{2} + 1$

Solve the following equations:

4. $5 + 2\sqrt{x} = 13$.
5. $7 + 4\sqrt{x} = 11$.
6. $\sqrt{3x} - 3 = 3$.
7. $\sqrt[3]{x - 1} = 3$.
8. $\sqrt{12 + x} = 6 - \sqrt{x}$.
9. $\sqrt{x + 1} = 2 + \sqrt{x - 7}$.
10. $\sqrt{10 - x} = \sqrt{22 - x} - 2$.
11. $\dfrac{5x - 1}{\sqrt{5x} + 1} = \dfrac{\sqrt{5x} + 1}{2}$.
12. $\sqrt{4x} - \sqrt{x} = 2$.
13. $\sqrt{9x} - \sqrt{0.25x} = 8$.
14. $\sqrt{x}(\sqrt{2} + 1) = 1$.
15. $\sqrt{2x} + \sqrt{x} = 1$.
16. $2\sqrt{x} - \sqrt{2x} = 2 + \sqrt{2}$.
17. $3\sqrt{x} + \sqrt{5x} = 3 + \sqrt{5}$.
18. $1 + \sqrt{x} = 2(1 - \sqrt{x})$.
19. $\dfrac{3x - 7}{\sqrt{3x} - \sqrt{7}} = \sqrt{7} + 3$.

208. Zero and Negative Exponents. Besides positive integral and fractional exponents there are zero and negative exponents.

It is easily seen that a^0 should mean 1. For since $a^m \cdot a^n = a^{m+n}$, then $a^0 \cdot a^n$ should equal a^{0+n} or a^n, or $1 \cdot a^n$.

It is also easily seen that a^{-n} should be defined to mean $\frac{1}{a^n}$. For since $a^m \cdot a^n = a^{m+n}$, then $a^{-n} \cdot a^n$ should equal $a^{-n+n} = a^0 = 1$. But this can be the case only if $a^{-n} = \frac{1}{a^n}$. Hence $a^{-1} = \frac{1}{a}$, $a^{-2} = \frac{1}{a^2}$, and so on.

209. Simplification of an Imaginary. Because $\sqrt{-a} = \sqrt{a(-1)} = \sqrt{a} \cdot \sqrt{-1}$ the imaginary expression $\sqrt{-a}$ may be written as the product of a real expression and $\sqrt{-1}$.

This is advisable in most of the operations with imaginaries.

210. Complex Number. The sum of a real number and an imaginary number is called a *complex number.*

Thus $2 + \sqrt{-3}$, or $2 + \sqrt{3}\sqrt{-1}$ is a complex number.

211. Product of Two Complex Numbers. Complex numbers are multiplied as here shown:

Since $\sqrt{-1} \cdot \sqrt{-1} = -1$, we see that

$3\sqrt{-1} \cdot (-2\sqrt{-1}) = -6 \cdot (-1) = 6$,

and $b\sqrt{-1} \cdot q\sqrt{-1} = bq \cdot (-1) = -bq$.

$$\begin{array}{r} 2 + 3\sqrt{-1} \\ 5 - 2\sqrt{-1} \\ \hline 10 + 15\sqrt{-1} \\ \quad - 4\sqrt{-1} + 6 \\ \hline 10 + 11\sqrt{-1} + 6 \\ = 16 + 11\sqrt{-1} \end{array}$$

This work suffices for checking roots in certain quadratic equations that will be met in the next chapter.

The topics of §§ 208–211 are treated more fully in Book II.

Exercise 119. Complex Numbers

All work written

Add the following:

1. $2 + 3\sqrt{-1}$, $4 - 2\sqrt{-1}$, $5 + 6\sqrt{-1}$, and $-7 - 4\sqrt{-1}$.
2. $3 - 6\sqrt{-1}$, $5 - 4\sqrt{-1}$, $2 - 3\sqrt{-4}$, and $-4 + 9\sqrt{-9}$.

Multiply as indicated:

3. $(4 + 2\sqrt{-1})(3 - 4\sqrt{-1})$.
4. $(2 + \sqrt{-4})(3 - \sqrt{-4})$.
5. $(2 + \sqrt{-9})(3 - \sqrt{-9})$.
6. $(4 + \frac{1}{2}\sqrt{-1})(4 - \frac{1}{2}\sqrt{-1})$

CHAPTER XVI

QUADRATIC EQUATIONS

212. Quadratic Equation. An equation which, when reduced to its simplest form, contains the second power, but no higher power, of an unknown quantity is called a *quadratic equation.*

For example, $x^2 = 9$, $x^2 - 3x = 0$, and $x^2 - 3x + 7 = 0$ are quadratic equations in x.

A quadratic equation may always be reduced to the type $ax^2+bx+c=0$, in which a is not zero, but b or c, or both b and c, may be zero.

Thus $3x^2 + 2x = 7$ is a quadratic equation in which $a = 3$, $b = 2$, and $c = -7$; $5x^2 - 2 = 0$ is a quadratic equation in which $a = 5$, $b = 0$, and $c = -2$. In $2x^2 - 3x = 0$, $a = 2$, $b = -3$, and $c = 0$; and in $4x^2 = 0$, $a = 4$, $b = 0$, and $c = 0$.

213. Coefficients of an Equation. In the equation $ax^2+bx+c=0$, a, b, and c are called the *coefficients of the equation.*

If a, b, and c are numbers expressed by figures, the equation is called a *numerical quadratic;* if some or all are represented by letters, the equation is called a *literal quadratic.*

Thus $x^2 + 7x - 3 = 0$ is a numerical quadratic, and $x^2 + ax + m = 5$ is a literal quadratic.

The term represented by c is called the *absolute term*, or the *constant term*, of the equation.

214. Affected and Pure Quadratics. A quadratic equation that contains both the second and first powers of the unknown quantity is called an *affected quadratic.*

If the first power of the unknown quantity is missing, the equation is called a *pure quadratic.*

Thus $x^2 - 5x + 6 = 0$ is an affected quadratic, and $x^2 - 4 = 0$ is a pure quadratic.

An affected quadratic is also called a *complete quadratic*, and a pure quadratic is also called an *incomplete quadratic.*

215. Solving a Pure Quadratic. We solve a pure quadratic as shown in the following examples:

1. Solve the equation $5x^2 + 15 = 3x^2 + 65$.

Given $5x^2 + 15 = 3x^2 + 65.$

Subtracting 15 and $3x^2$, $2x^2 = 50.$

Dividing by 2, $x^2 = 25.$

Extracting the square root, $x = \pm 5.$

Check. Substitute either $+5$ or -5 for x in the *given* equation.

Then $5 \cdot 25 + 15 = 3 \cdot 25 + 65,$

or $140 = 140.$

When we extracted the square root of x^2 and 25 we might have written the result $\pm x = \pm 5$. In this case we should then have

$$+x = +5, \quad +x = -5, \quad -x = +5, \quad -x = -5.$$

But since we wish only the values of $+x$, we should obtain from $-x = +5$ the value $x = -5$, and from $-x = -5$ the value $x = 5$, which we have already. Therefore *it is unnecessary to write the sign* $\pm$ *before both members.*

Hence there are only two roots, both being rational.

2. Solve the equation $x^2 + 7 = 14$.

Given $x^2 + 7 = 14.$

Subtracting 7, $x^2 = 7.$

Extracting the square root, $x = \pm\sqrt{7}.$

Here the two roots are real but both are irrational.

Check. $(\pm\sqrt{7})^2 + 7 = 7 + 7 = 14.$

3. Solve the equation $x^2 + 5 = 0$.

Given $x^2 + 5 = 0.$

Subtracting 5, $x^2 = -5.$

Extracting the square root, $x = \pm\sqrt{-5}.$

Here the two roots are imaginary. We may show that they are correct by substituting in the given equation.

4. Solve the equation $x^2 + 9 = 9$.

Given $x^2 + 9 = 9.$

Subtracting 9, $x^2 = 0.$

Extracting the square root, $x = 0.$

Of course we may write ± 0 instead of 0, if we wish, the value being the same. That is, the roots in this example are both zero.

Exercise 120. Pure Quadratics

Examples 1 to 5, oral — Examples 6 to 19, written

1. Solve the equation $x^2 = 25$, and check.
2. Solve the equations $x^2 = 9$ and $x^2 - 9 = 0$.
3. Solve the equations $x^2 - 36 = 0$ and $x^2 - 64 = 0$.

Solve the following equations:

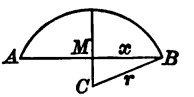
4. $x^2 - 81 = 0$.
5. $x^2 - 121 = 0$.
6. $4x^2 - 196 = 0$.
7. $11x^2 - 176 = 0$.
8. $x^2 - a^2 = 0$.
9. $x^2 - a = 0$.
10. $x^2 + a = 0$.
11. $5t^2 + 7 = 3t^2 + 25$.

Solve the following equations, and check the roots:

12. $\dfrac{35 + 3x}{x + 1} = \dfrac{x - 55}{3x - 53}$.
13. $\dfrac{x - 2}{3x + 14} = \dfrac{24 - 3x}{28 - x}$.
14. $\dfrac{x^2 + 5x + 7}{x + 1} = \dfrac{x^2 + x + 1}{x - 1}$.
15. $\dfrac{a + x}{b + x} = \dfrac{x - a}{b - x}$.

16. An arch 6 ft. high is described with a radius 10 ft. Find the length of the span.

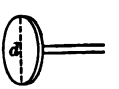
Let $MB = x$. What is the length of MC? Find the value of x; of $2x$.



17. A cistern with a square base is 8 ft. deep. When full it contains 7260 gallons. Allowing $7\frac{1}{2}$ gallons to the cubic foot, find the side of the base.

18. What must be the diameter d of the piston shown in the figure below, to have a steam pressure of 100 lb. per square inch exert a total pressure on the piston of 3850 lb.?

Use $\frac{22}{7}$ for π in this example.



19. Two men start from the same place at the same time. One walks east at the rate of 3 mi. an hour and the other walks north at the rate of 4 mi. an hour. How soon will they be 15 mi. apart?

216. Solving the Affected Quadratic by Factoring. The method of solving by factoring will be understood from a single example. Required to solve the equation $x^2 - 5x + 6 = 0$.

Given $x^2 - 5x + 6 = 0.$

Factoring, $(x - 2)(x - 3) = 0.$

Since the product of $x - 2$ and $x - 3$ equals 0, one or the other of these factors must equal 0, because no two numbers can have 0 for a product unless one of them is 0. Furthermore, it makes no difference which factor is 0, since 0 multiplied by any other number is 0.

If $x - 2 = 0,$

$x = 2;$

and if $x - 3 = 0,$

$x = 3.$

Therefore x may equal 2 or it may equal 3. Since there can be only two factors of the polynomial, there can be only two roots.

Check. Substituting 2 in the given equation,

$$2^2 - 5 \cdot 2 + 6 = 4 - 10 + 6 = 0.$$

Substituting 3, $3^2 - 5 \cdot 3 + 6 = 9 - 15 + 6 = 0.$

Exercise 121. Solving by Factoring

Examples 1 to 14, oral — Examples 15 to 32, written

1. Factor $x^2 - 6x + 8$; $x^2 - 7x + 10$; $x^2 - 8x + 15$.
2. Factor $x^2 - 10x + 21$; $x^2 - 9x + 20$; $x^2 - 10x + 24$.
3. Factor $x^2 - 11x + 28$; $x^2 - 11x + 30$; $x^2 - 12x + 35$.
4. Solve the equation $(x - 1)(x - 2) = 0$.
5. Solve the equation $x^2 - 3x + 2 = 0$.
6. Solve the equation $(x - 2)(x + 3) = 0$.

Solve the following equations:

7. $x(x - 2) = 0.$
8. $x(x + 7) = 0.$
9. $x(x - 3) = 0.$
10. $x^2 - 3x = 0.$
11. $(x - 1)(x - 5) = 0.$
12. $(x - 1)(x + 5) = 0.$
13. $(x + 2)(x - 9) = 0.$
14. $(x - 2)(x - 9) = 0.$

Solve the following equations:

15. $x^2 - 7x + 12 = 0$.
16. $x^2 - 11x + 30 = 0$.
17. $x^2 - 12x + 35 = 0$.
18. $x^2 + 45 = 14x$.
19. $x^2 + 50 = 15x$.
20. $x^2 + 78 = 19x$.
21. $x^2 + 4x - 45 = 0$.
22. $x^2 + 5x - 50 = 0$.
23. $y^2 + 4y = 77$.
24. $u^2 - 7u = 60$.
25. $t^2 = 7t + 78$.
26. $6x^2 + x = 1$.

27. The area of a rectangle is 12 sq. ft. The width is 1 ft. less than the length. Find the dimensions.

Which root is it reasonable to take?

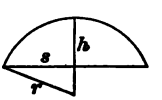
28. The area of a rectangle is 45 sq. in. The length is 4 in. greater than the width. Find the dimensions.

29. A man purchased part of a sheet of 2-cent postage stamps for $1.60. When he was folding it he saw that there were two more rows on one side than on the other. How many stamps were there on each side?

30. If a bar of iron weighing 80 lb. were drawn out 1 ft. longer, it would weigh $6\frac{2}{3}$ lb. less per linear foot. How long is it?

If it is x feet long, what is its present weight per foot? If it were 1 ft. longer, what would be its weight per foot? How do these compare? Which root is it reasonable to take?

31. In building a circular arch the radius is found by the formula $r = \frac{s^2 + h^2}{2h}$, in which $r =$ the radius of the circle, $s =$ half the span of the arch, and $h =$ the height of the arch. If r is known to be 10, and the span $(2s)$ to be 16, find the value of h.



Which root is it reasonable to take?

32. A tank that has two pipes can be filled in 2 hr. less time by one pipe than by the other, and by both together in 2 hr. 55 min. How long will it take each pipe to fill the tank?

217. Completing the Square. A second method of solving the quadratic equation, and one commonly used, is that called the solution by *completing the square.*

Since $(x+a)^2 = x^2 + 2ax + a^2$, the binomial $x^2 + 2ax$ lacks only a^2 of being a perfect square. This a^2 is the square of half the coefficient of x. Therefore if we have an equation in the form $x^2 + 2ax = b$, we may make the first member a perfect square (*complete the square*) by adding a^2 to both members.

For example, given $x^2 + 6x = 55.$

Adding $(\frac{6}{2})^2$, or 3^2, $x^2 + 6x + 9 = 64.$

Extracting the square root, $x + 3 = \pm 8.$

$\therefore x = -3 \pm 8$

$= 5 \text{ or } -11.$

ax	a^2
x^2	ax
x	a

Check. $5^2 + 6 \cdot 5 = 25 + 30 = 55.$

$(-11)^2 + 6 \cdot (-11) = 121 - 66 = 55.$

If the equation is in the form $x^2 - 2ax = b$, we may evidently complete the square in the same way, by adding a^2, the square of $-a$, to both members. We then have

$$x^2 - 2ax + a^2 = b + a^2;$$

whence $$x - a = \pm\sqrt{b + a^2},$$

and $$x = a \pm \sqrt{b + a^2}.$$

If the equation has a coefficient for x^2 other than 1, we may divide both members by this coefficient and reduce the equation to the form $x^2 + 2ax = b$.

For example, given $2x^2 - 3x = 9.$

Dividing by 2, $x^2 - \frac{3}{2}x = \frac{9}{2},$

or $x^2 - 2 \cdot \frac{3}{4}x = \frac{9}{2},$

in which $-\frac{3}{4}$ equals the a of the equation $x^2 + 2ax = b$.

Therefore to solve an affected quadratic:

1. *Reduce the equation to the form* $x^2 + 2ax = b$.
2. *Add to each member the square of half the coefficient of* x.
3. *Extract the square root of each member of the resulting equation.*
4. *Solve the two resulting simple equations.*

1. Solve the equation $2x^2 - 3x = 9$.

Given $2x^2 - 3x = 9.$

Dividing by 2, $x^2 - \frac{3}{2}x = \frac{9}{2}.$

Adding the square of $\frac{1}{2} \cdot \frac{3}{2}$; that is, adding $(\frac{3}{4})^2$, or $\frac{9}{16}$,

$$x^2 - \tfrac{3}{2}x + \tfrac{9}{16} = \tfrac{9}{2} + \tfrac{9}{16}$$
$$= \tfrac{81}{16}.$$

Extracting the square root, $x - \frac{3}{4} = \pm \frac{9}{4}.$

Solving for x, $x = \frac{3}{4} \pm \frac{9}{4}$
$= 3$ or $-\frac{3}{2}$.

Check. $2 \cdot 3^2 - 3 \cdot 3 = 18 - 9 = 9.$

$2 \cdot (-\frac{3}{2})^2 - 3(-\frac{3}{2}) = \frac{9}{2} + \frac{9}{2} = 9.$

In solving this and similar equations we may multiply both members by 2 and have

$$(2x)^2 - 3(2x) = 18;$$

whence $(2x)^2 - 3(2x) + \frac{9}{4} = \frac{81}{4}$, and $2x - \frac{3}{2} = \pm \frac{9}{2}$; whence $x = 3$ or $-\frac{3}{2}$ as before.

2. Solve the equation $x + \dfrac{2}{x} = 4.$

Given $x + \frac{2}{x} = 4.$

Multiplying by x, $x^2 + 2 = 4x.$

Subtracting 2 and $4x$, $x^2 - 4x = -2.$

Adding $(\frac{4}{2})^2$, or 2^2, or 4, $x^2 - 4x + 4 = 2.$

Extracting the square root, $x - 2 = \pm\sqrt{2}.$

Solving for x, $x = 2 \pm \sqrt{2}.$

Check. Substituting $2 + \sqrt{2}$ for x in the original equation,

$$2 + \sqrt{2} + \frac{2}{2 + \sqrt{2}} = 4.$$

Rationalizing the denominator of the fraction,

$$2 + \sqrt{2} + \frac{2(2 - \sqrt{2})}{4 - 2} = 4,$$

or

$$2 + \sqrt{2} + 2 - \sqrt{2} = 4.$$

In the same way we may check for $2 - \sqrt{2}$.

If the practical problem in which such an equation enters requires the square root extracted to two or more decimal places, this should be done. The results would then be $2 \pm 1.414+$, or $3.414+$ and $0.586-$.

Exercise 122. Affected Quadratics

Examples 1 to 8, oral — Examples 9 to 40, written

1. What must be added to $x^2 + 8x$ to complete the square?
2. What must be added to $x^2 + 6x$ to complete the square?

Complete the squares in the following cases:

3. $x^2 - 4x$.
4. $x^2 - 6x$.
5. $x^2 + x$.
6. $x^2 - x$.
7. $x^2 + 5x$.
8. $x^2 - 7x$.

Solve the following equations:

9. $x^2 - 20x - 9 = 60$.
10. $x^2 - 20x - 6 = 0$.
11. $x^2 - 12x - 2 = 26$.
12. $x^2 + 8x - 9 = 200$.
13. $x^2 - 4x - 7 = 110$.
14. $x^2 + 5x - 4 = 100$.
15. $x^2 - 8x - 5 = 100$.
16. $x^2 + x - 82 = 100$.
17. $x^2 + 35x + 300 = 0$.
18. $x^2 + x + 1 = 0$.
19. $x^2 - x + 1 = 0$.
20. $x^2 + 3x + 2 = 0$.
21. $x^2 - 7x + 5 = 0$.
22. $x^2 - 9x + 7 = 0$.
23. $2x^2 - 8x - 3 = 0$.
24. $3x^2 - 6x + 10 = 0$.
25. $5x^2 - 2x + 30 = 0$.
26. $7x^2 - 28x + 3\frac{1}{2} = 0$.

27. Find the value of p in the equation $p^2 - 7p = 6$.

28. Find the value of t in the equation $2t^2 - 3t = 35$.

29. Find the value of v in the equation $v^2 + 11 = 8v$.

30. Find to two decimal places the value of K in the equation $K(14 - K) = 47$.

31. Find to three decimal places the value of P in the equation $P^2 = 6(3P - 13)$.

32. Solve the equation $\dfrac{5x - 7}{9} + \dfrac{14}{2x - 3} = x - 1$.

33. Solve the equation $\dfrac{6x + 4}{5} - \dfrac{15 - 2x}{x - 3} = \dfrac{7(x - 1)}{5}$.

34. Solve the equation $\dfrac{x - 5}{x + 3} + \dfrac{x - 8}{x - 3} = \dfrac{3(1 - x)}{x^2 - 9}$.

35. If from a square piece of paper I cut a strip 2 in. wide as is shown in the figure, the area of the rest of the paper is 63 sq. in. What is the side of the square?

Let x = the number of inches in the side.
Then x^2 = the area of the square in square inches.
Then $x^2 - 2x = 63$.
Solving, $x = 9$ or -7.

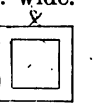
Of these roots -7 is inadmissible by the conditions of the problem, although it satisfies the algebraic equation. Usually only one of the algebraic roots meets the conditions of the problem.

36. A circle with radius 7 in. is cut from a square as shown in the figure. The number of square units of area remaining equals six more than twelve times the side of the square. Find the side of the square. (Use $\frac{22}{7}$ for π.)

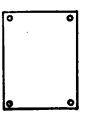
37. A certain positive number multiplied by the sum of itself and 9 equals 220. What is the number? If the problem specified a certain negative number, what would be the result?

38. If twice a certain integer should be subtracted from 7, and the remainder multiplied by the integer, the product would be 6. Find the integer. If the problem specified a fraction instead of an integer, what would be the result? If the problem placed no limitation on the kind of number, what would be the result?

39. The molding for a square picture frame is 2 in. wide. The area of the opening for the picture is 54 sq. in. Find the outside dimensions of the frame. Are both results admissible?



40. A steel plate is 3 in. longer than it is wide. In the four corners holes are drilled for rivets, the area of each opening being 1 sq. in. The area of the rest of the rectangle is 454 sq. in. Find the dimensions of the rectangle. Are both results admissible?



218. Literal Quadratics. A quadratic with literal coefficients may be solved by completing the square, as shown in the following examples:

1. Solve the equation $x^2 + bx = c$.

Given $x^2 + bx = c$.

Adding $\left(\frac{b}{2}\right)^2$, $x^2 + bx + \frac{b^2}{4} = c + \frac{b^2}{4}$.

Extracting the square root, $x + \frac{b}{2} = \pm\sqrt{c + \frac{b^2}{4}}$

$= \pm \frac{1}{2}\sqrt{4c + b^2}$.

Subtracting $\frac{b}{2}$, $x = -\frac{b}{2} \pm \frac{1}{2}\sqrt{4c + b^2}$.

The amount of checking to be required in the case of such complicated results must be determined by the teacher according to the needs of the class.

2. Solve the equation $(x + a)^2 + 4(x + a) = 21$.

Consider this as a quadratic in $x + a$, and add 4.

Then $(x + a)^2 + 4(x + a) + 4 = 25$.

Extracting the square root, $x + a + 2 = \pm 5$.

Therefore $x = -a + 3$ or $-a - 7$.

Exercise 123. Literal Quadratics

Examples 1 to 3, oral — Examples 4 to 53, written

1. Solve the equations $x^2 = a$; $ax^2 = b$; $a^2x^2 = b^2$.

2. Solve the equations $x(x - a) = 0$; $x(x^2 - a^2) = 0$.

3. Solve the equations $2x(x - 1) = 0$; $ax(x - 1) = 0$.

Solve the following equations:

4. $x^2 + px = 6p^2$.

5. $x^2 - px = 24p^2$.

6. $x^2 + 5ax = 6a^2$.

7. $x^2 + b^2 = ax$.

8. $x^2 - ax = b$.

9. $x^2 + 3cx - 10c^2 = 0$.

10. $x^2 = 5m(10m - x)$.

11. $2x^2 + 3kx = 119k^2$.

12. $x^2 + mx = m^2n(n + 1)$

13. $x^2 = abc(2abc - x)$.

Solve the following equations:

14. $x^2 = (a - b)\,x + ab.$

15. $mx^2 = p + 2\,nx.$

16. $ax^2 + b = 2\,ax + b^2.$

17. $ax^2 + 5 = 4\,ax + 7.$

18. $px^2 + 2\,px = q + r.$

19. $y^2 + 4\,aby = 5\,a^2b^2.$

20. $u^2 + 6\,mu = 16\,m^2.$

21. $w^2 - 8\,cw = -16\,c^2.$

22. $at^2 - a^2bt = 0.$

23. $t^2 - at = a^2k\,(k - 1).$

24. $v^2 - bv = 1 + b.$

25. $2\,v^2 + 2\,bv = 4 - b^2.$

26. $x^2 = k^2\,(2\,k^2 - x).$

27. $w\,(w + 2\,b) = 24\,b^2.$

28. $w^2 - (a - b)\,w = 2\,b\,(a + b).$

29. $v\,(v + al) = l\,(a + 1).$

30. $4\,v\,(v + 4) = l\,(l + 8).$

31. $x^2 + (a + b)\,x = 2\,a\,(a + b).$

32. $9\,x^2 + 27\,ax - 10\,a^2 = 0.$

33. $a^2x^2 + a^2x = 9 + 3\,a^2.$

34. $6\,x^2 - a^2 + 5\,ax = 0.$

35. $x^2 + x\sqrt{a} = 6\,a.$

36. $x^2 + 2\,x\sqrt{ab} = 15\,ab.$

37. $4\,x\left(x + 2\sqrt{-a}\right) + 5\,a = 0.$

38. Solve the equation $(a + x)^2 + 2\,(a + x) = 35$, and check the roots. What are the values of the roots if $a = 1$? if $a = 2$?

Solve the following equations:

39. $\dfrac{a}{x} + \dfrac{x}{a} = 1.$

40. $\dfrac{a}{2\,x} + \dfrac{x}{2\,a} = -1.$

41. $x + \dfrac{a}{x} = b.$

42. $\dfrac{x}{a} + a = \dfrac{1}{x}.$

43. $x - \dfrac{m}{x} = n.$

44. $x + \dfrac{1}{x} + a = b.$

45. $x - \dfrac{1}{x} - a = b.$

46. $x + \dfrac{a}{x} + b = a.$

47. $x - \dfrac{a}{x} - b = a.$

48. $ax + \dfrac{b}{x} + c = d.$

49. $x^2 - (m + n)\,x + (m + p)\,(n - p) = 0.$

50. $x^2 - (a - b)\,x - (a - 1)\,(b - 1) = 0.$

51. $x^2 + 2\,ab\,(a^2 + b^2) = (a + b)^2x.$

52. $(a + b + c)\,x^2 - (2\,a + b + c)\,x + a = 0.$

53. $(a - x)^2 + (b - x)^2 = 2.5\,(a - x)\,(b - x).$

Exercise 124. Radical Equations

Examples 1 to 4, oral — Examples 5 to 24, written

1. Solve the equation $2\sqrt{x} = 5$. Check the result

2. Solve the equation $\sqrt{x+7} = 4$. Check the result.

3. Solve the equation $\sqrt{\frac{1}{2}x+3} = 5$. Check the result.

4. Solve the equation $x - 3\sqrt{x} + 2 = 0$ as if the unknown quantity were $\sqrt{x}$. Then find the value of x.

Solve the following equations:

5. $x + 5 = 5\sqrt{x-1}$.

6. $\sqrt{x+\sqrt{x+4}} = 2\sqrt{2}$.

7. $\sqrt{x+\sqrt{x+2}} = 2\sqrt{\frac{5}{2}}$.

8. $\sqrt{\sqrt{x-2}+x} = 2$.

9. $\sqrt{x+\sqrt{x+3}+1} = 0$.

10. $\sqrt{x+\sqrt{x+9}} = \sqrt{-3}$.

11. $\sqrt{x+7+\sqrt{5(x-2)}} = 3$.

12. $\sqrt{x+3}+\sqrt{x+8} = 5\sqrt{x}$.

13. $2\sqrt{5+2x}-\sqrt{13-6x} = \sqrt{37-6x}$.

14. $\sqrt{2x+1}+\sqrt{x-3} = 2\sqrt{x}$.

15. $\sqrt{5x-1}-\sqrt{8-2x} = \sqrt{x-1}$.

16. $\sqrt{a-x}+\sqrt{b-x} = \sqrt{a+b-2x}$.

17. $\sqrt{(a+x)(x+b)}+\sqrt{(a-x)(x-b)} = 2\sqrt{ax}$.

18. $2x^2 + a\sqrt{b(b+4x)} = a(b+2x)$.

19. $\sqrt{2a-b+2x}-\sqrt{10a-9b-6x} = 4\sqrt{a-b}$.

20. $x+\sqrt{a^2+x^2} = \dfrac{5a^2}{\sqrt{a^2+x^2}}$.

22. $\dfrac{\sqrt{1+x}+\sqrt{x-7}}{\sqrt{1+x}-\sqrt{x-7}} = 2$.

21. $\sqrt{\dfrac{x}{a}}+\sqrt{\dfrac{a}{x}} = \sqrt{\dfrac{x}{b}}+\sqrt{\dfrac{b}{x}}$.

23. $12\sqrt{\dfrac{x}{2}}+5\sqrt{\dfrac{2}{x}} = 26.5$.

24. $\dfrac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}-\dfrac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}} = 8\sqrt{x^2-1}$.

Exercise 125. Problems involving Quadratics

Examples 1 to 5, oral — Examples 6 to 28, written

1. State an equation showing that the product of two consecutive numbers is 182.

2. State an equation showing that if 3 is taken from a certain number the result equals 28 divided by the number.

3. State an equation showing that the difference between the cubes of two consecutive numbers is 37.

4. State an equation showing that the sum of a number and its reciprocal is $1\frac{4}{12}$.

5. State an equation showing that the sum of the squares of two consecutive numbers is 145; that the sum of the squares of two consecutive even numbers is 100; that the sum of the squares of two consecutive odd numbers is 130.

6. Solve Exs. 1–4.

7. Solve the three problems in Ex. 5.

8. Find a number such that its square is four times the product of the number and 12.

9. Find a number such that its square is four times the sum of the number and 8.

10. Find two consecutive numbers the sum of whose squares is 313.

11. Find two consecutive even numbers the sum of whose squares is 884.

12. The sum of two adjacent sides of a rectangle is 23 in. and the area of the rectangle is 120 sq. in. Find the dimensions.

13. The perimeter of a rectangle is 42 in. and the area is 108 sq. in. Find the dimensions.

14. One side of a rectangle is 5 in. shorter than the other, and the area is 176 sq. in. Find the dimensions.

15. One side of a rectangle is $3\frac{1}{2}$ in. longer than the other, and the area is 200 sq. in. Find the dimensions.

A straight line AB, 1 inch long, is divided by the point P so as to satisfy one of the following conditions. Find the length of AP in each case:

16. $\overline{AP}^2 = AB \cdot PB$.

17. $\overline{AP}^2 = 2\,AB \cdot PB$.

18. $2\,\overline{AP}^2 = 3\,AB \cdot PB$.

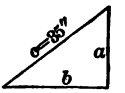
19. $\overline{AP}^2 = 2\,\overline{PB}^2$.

20. $\overline{AP}^2 = \overline{AB}^2 - 2\,\overline{PB}^2$.

21. $\overline{AP}^2 - \overline{PB}^2 = \frac{1}{4}$ sq. in.

22. In the right triangle shown below, the hypotenuse is 35 in. and the side a is 7 in. shorter than the side b. Find the length of each side.

23. In the same figure, if a is 10 in. longer than b, and c is 50 in., what is the length of each side?



24. In the same figure, if a is 33 ft. longer than b, and c is 165 ft., what is the length of each side?

25. A lady planted a square bed of flowers next to her house. Each year the plants spread 3 in. on each of the three open sides. At the end of three years the bed had an area of 4012 sq. in. What was the length of the original bed?

26. A rectangular plot of grass has a perimeter of 232 ft. The length is 4 ft. greater than the width. A walk surrounds the plot, and the area of the walk equals the area of the plot. What is the width of the walk?

27. A cylindric tomato can has its height 1 in. greater than the diameter of the base. To make such a can requires 46 sq. in. of tin, 2 sq. in. of which are lost in overlapping. Taking π as $3\frac{1}{7}$, find the diameter of the base.

28. It is shown in physics that if a body is thrown downward with an initial velocity of V feet per second, the distance d feet that it will pass through in t seconds is given by the formula $d = Vt + 16\,t^2$. If a body is thrown downward from a height of 1900 ft. with an initial velocity of 60 ft. per second, how long will it take it to reach the ground?

CHAPTER XVII

SIMULTANEOUS QUADRATIC EQUATIONS

219. Simultaneous General Quadratics. The most general type of quadratic equation with two unknown quantities contains both of these quantities in the first and second degrees, contains their product, has an absolute term, and has coefficients for all the terms containing an unknown quantity.

That is, just as $ax^2 + bx + c = 0$ is the most general quadratic equation with one unknown quantity, so $ax^2 + bxy + cy^2 + dx + ey + f = 0$ is the most general quadratic equation with two unknown quantities.

Two such equations may be represented by

$$ax^2 + bxy + cy^2 + dx + ey + f = 0,$$

and $$a'x^2 + b'xy + c'y^2 + d'x + e'y + f' = 0.$$

If we eliminated y, we should have an equation of the fourth degree in x that could not be solved by quadratics.

Therefore, *in general, two simultaneous quadratic equations cannot be solved by the method of quadratics.*

There are, however, certain special types that can be solved, and these are considered in this chapter.

That the student may see the difficulties, even in the case of two quadratic equations that seem easy of treatment, consider the system

$$x^2 + y = 7, \tag{1}$$

$$x + y^2 = 11. \tag{2}$$

From (1), $$y = 7 - x^2. \tag{3}$$

Substituting (3) in (2), $$x + (7 - x^2)^2 = 11,$$

whence $$x^4 - 14x^2 + x + 38 = 0.$$

This is an equation of the fourth degree, and although one root ($x = 2$) may easily be found by factoring, it cannot be obtained from this equation by the methods of quadratics.

220. Simple and Quadratic Equations. Before considering two simultaneous quadratic equations we shall consider two simultaneous equations of which one is simple and the other quadratic.

Solve the equations

$$2x^2 + y^2 = 33 \quad (1)$$

$$2x + y = 9 \quad (2)$$

Subtracting $2x$ from (2), $y = 9 - 2x$. (3)

Substituting (3) in (1), $2x^2 + (9 - 2x)^2 = 33$,

whence $x^2 - 6x + 8 = 0$.

Solving, $x = 2$ or 4.

Substituting in (3), $y = 5$ or 1.

We therefore have two pairs of roots, in which

$x = 2$ when $y = 5$,

and $x = 4$ when $y = 1$.

For the first pair

equation (1) becomes $2 \cdot 2^2 + 5^2 = 8 + 25 = 33$,

equation (2) becomes $2 \cdot 2 + 5 = 4 + 5 = 9$,

and similarly for the second pair.

In substituting, join with x the corresponding value of y.

That is, $x = 2$ will not check with $y = 1$, nor $x = 4$ with $y = 5$.

Exercise 126. Simple and Quadratic Equations

Examples 1 and 2, oral — Examples 3 to 8, written

1. Solve the equations $x = y$, $x^2 + y^2 = 2$.
2. Solve the equations $x = -y$, $x^2 + y^2 = 2$.

Solve the following equations:

3. $3x^2 + y^2 = 43$
   $5x - y = 11$
4. $2x^2 + y^2 = 51$
   $\frac{1}{2}x = -y$
5. $xy = -15$
   $4x + 7y = 23$
6. $3x - y = 5$
   $xy - x = 0$
7. $x^2 - xy + y^2 = 7$
   $2x - 3y = 0$
8. $x^2 - xy - 2y^2 = 7$
   $x - y - 3 = 0$

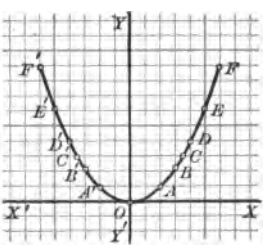
221. Graph of a Quadratic Equation. To understand more clearly the relation of a linear to a quadratic equation it is helpful to study the graph of each. Since we have already studied the graphs of linear equations, it is necessary at this time to consider only the graph of the quadratic.

1. Plot the equation $x^2 = 4y$.

If y is negative, x is imaginary and there are no points. We therefore take positive values of y.

If $y =$	0	1	2	3	4	$6\frac{1}{4}$	9
Then $x =$	0	± 2	± 2.8	± 3.5	± 4	± 5	± 6
Points $=$	O	A, A'	B, B'	C, C'	D, D'	E, E'	F, F'

Plot the several points A, A', B, B', $\cdots$. Through these draw the smooth curve from O to F and F'. This curve is the graph of the equation $x^2 = 4y$. Of course we might take larger values for y and find the corresponding values for x, it being evident that the curve is infinite in extent.



This curve is called a *parabola*. The graph of a quadratic equation containing the first power of one variable and the second power with or without the first power of the other variable is always a parabola. In particular, $x^2 = ay$ and $y^2 = bx$ are both parabolas. If we should plot the latter equation ($y^2 = bx$), we would find that it lies to the right of YY', in the first and fourth quadrants.

In the equation $x^2 = 4y$, when $y = 0$, $x^2 = 0$. The two values of x are then both equal to 0. We measure the values of x on the axis of x. If, therefore, we have the equation $x^2 = 0$, the two roots are equal, and each is 0.

In the same equation ($x^2 = 4y$), when $y = 1$, $x^2 = 4$. The two values of x are then ± 2, and are the abscissas of the points $(2, 1)$, $(-2, 1)$. Similarly, we can solve and find the two values of x corresponding to any value of y by simply measuring the abscissas. By the help of coördinate paper this is easily done.

2. Plot the equation $x^2 + y^2 = 25$.

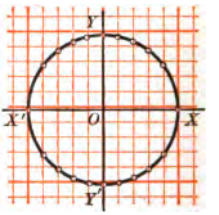
We may write this $y^2 = 25 - x^2$.

If $x^2 > 25$, or if $x > 5$, y is imaginary, and similarly if $x < -5$. We will first take for x values from -4 to $+4$.

If $x =$	-4	-3	-2	-1	0	1	2	3	4
Then $y =$	± 3	± 4	± 4.6	± 4.9	± 5	± 4.9	± 4.6	± 4	± 3

Furthermore, if $x = \pm 5$, $y = 0$, which gives two other points.

Plotting the points and drawing the smooth curve as before, the graph is seen to be a *circle* (circumference). The graph of a quadratic equation in the form $x^2 + y^2 = r^2$ is always a circle.



There is a more general form for the equation of a circle, $x^2 + y^2 + ax + by + c = 0$, where a, b, and c are any given numbers.

3. Plot the equation $4x^2 + 9y^2 = 288$.

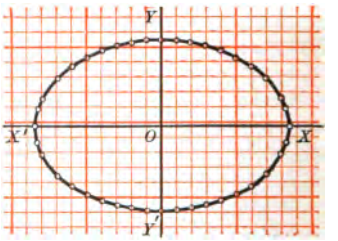
We may write this $y^2 = 32 - \frac{4}{9}x^2$.

If $\frac{4}{9}x^2 > 32$, or if $\frac{2}{3}x > \sqrt{32}$, or $\frac{2}{3}x < -\sqrt{32}$, y is imaginary; that is, x cannot be greater than $\frac{3}{2}\sqrt{32}$, or about 8.5, and x cannot be less than -8.5.

If $x =$	± 8.3	± 8	± 7	± 6	± 5	± 4	± 3	± 2	± 1
Then $y =$	± 1.2	± 1.9	± 3.2	± 4	± 4.6	± 5.0	± 5.3	± 5.5	± 5.6

Furthermore, if $x = \pm 8.5$, $y = 0$; and if $x = 0$, $y = \pm 5.7$.

It will be seen that $+8.3$ corresponds both to $+1.2$ and to -1.2, and that -8.3 also corresponds both to $+1.2$ and to -1.2.



Plotting the points and drawing the curve, the graph is an *ellipse*. The graph of a quadratic equation in the form $ax^2 + by^2 = c$ is always an ellipse, a, b, and c having any values, and a and b having like signs. For example, $2x^2 + 3y^2 - 18 = 0$ is the equation of an ellipse.

4. Plot the equation $xy = 5$.

We may write this $y = \frac{5}{x}$. We then prepare these two tables:

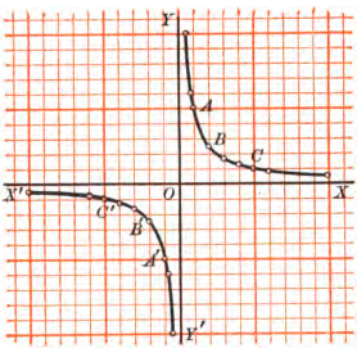
If $x =$	$\frac{1}{2}$	$\frac{5}{6}$	1	2	3	4	5	6	10
Then $y =$	10	6	5	$2\frac{1}{2}$	$1\frac{2}{3}$	$1\frac{1}{4}$	1	$\frac{5}{6}$	$\frac{1}{2}$

If $x =$	$-\frac{1}{2}$	$-\frac{5}{6}$	-1	-2	-3	-4	-5	-6	-10
Then $y =$	-10	-6	-5	$-2\frac{1}{2}$	$-1\frac{2}{3}$	$-1\frac{1}{4}$	-1	$-\frac{5}{6}$	$-\frac{1}{2}$

The graph is in this case an *hyperbola*. An equation in the form $xy = a$ is always the equation of an hyperbola.

The three curves already mentioned — the parabola, ellipse, and hyperbola — are called *conic sections* or *conics*. The circle may be considered a special form of the ellipse.

The graph of a quadratic equation is always a conic or a pair of straight lines.

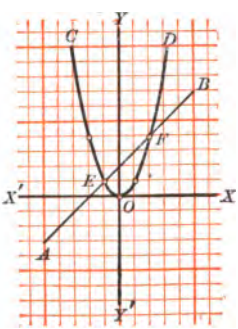


222. Graph of a Linear and Quadratic Equation. We see therefore that the roots of a linear and quadratic equation may be found by the intersection of graphs.

For example, solve the equations

$$x^2 = y$$
$$x - y + 2 = 0$$

Here the graph of $x^2 = y$ is the parabola CD, and the graph of $x - y + 2 = 0$ is a straight line AB. Their common points, E and F, are $(-1, 1)$ and $(2, 4)$. Hence the common values of x and y are $x = -1, +2$, $y = 1, +4$.



Exercise 127. Graphs

Examples 1 to 5, oral — Examples 6 to 29, written

1. What kind of graph has the equation $x + 2y = 7$?
2. What kind of curve is represented by $x^2 + y^2 = 16$?
3. What kind of conic is represented by $y^2 = 10x$?
4. What kind of conic is represented by $x^2 + 5y^2 = 16$?
5. What kind of conic is represented by $xy = 25$?

Plot the following equations:

6. $x^2 + y^2 = 9$.
7. $x^2 + 4y^2 = 9$.
8. $x^2 = 9y$.
9. $x^2 + 4x + 3 = y$.
10. $x^2 + 5x + 4 = y$.
11. $xy = 4$.
12. $x^2 - 5x + 6 = y$.
13. $x^2 + 2y^2 = 16$.

Find by graphs the roots of the following by letting the first member equal y, and then considering the graph when $y = 0$:

14. $x^2 - 7x + 12 = 0$.
15. $x^2 - 9x + 20 = 0$.
16. $x^2 - 13x + 42 = 0$.
17. $x^2 + 3x - 10 = 0$.
18. $x^2 + 4x - 21 = 0$.
19. $x^2 - 4x - 21 = 0$.

Solve exactly or approximately by the use of graphs:

20. $x^2 + y^2 = 169$
    $3x - 2y = -9$
21. $x^2 + y^2 = 100$
    $5x + y = 46$
22. $x^2 + y^2 = 100$
    $3x + 4y = 50$
23. $x^2 + 9y^2 = 81$
    $x - 3y = 5$
24. $2x + y = 5$
    $3x^2 - 7y^2 = 5$
25. $xy = 6$
    $x - y = 0$
26. $x^2 = 4y$
    $x + 2y = 6$
27. $xy = 24$
    $x + 2y = 14$
28. $x^2 = 9y$
    $2x + y = 16$
29. $x^2 + x + 3 = y$
    $x - 2y + 4 = 0$

223. Two Quadratics, one being Homogeneous. Given two simultaneous quadratics, in one of which the terms are all of the second degree, a solution is always possible.

Solve the equations

$$x^2 + xy - 6y^2 = 0 \qquad (1)$$

$$x^2 + 4x + 3y = 69 \qquad (2)$$

From (1), $x = \dfrac{-y \pm \sqrt{y^2 + 24y^2}}{2} = \dfrac{-y \pm 5y}{2} = 2y$ or $-3y$.

Substituting $2y$ for x in (2),

$4y^2 + 8y + 3y = 69$,

or $4y^2 + 11y = 69$.

Solving, $y = 3$ or $-2\frac{3}{4}$.

$\therefore x = 2y = 6$ or $-2\frac{3}{2}$.

Substituting $-3y$ for x in (2),

$9y^2 - 12y + 3y = 69$,

or $y^2 - y = 2\frac{3}{3}$.

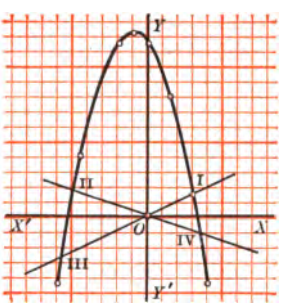
Solving, $y = \dfrac{3 \pm \sqrt{285}}{6}$.

$\therefore x = -3y = \dfrac{-3 \mp \sqrt{285}}{2}$.

Therefore, when $x = 6, \dfrac{-3-\sqrt{285}}{2}, -\dfrac{23}{2}, \dfrac{-3+\sqrt{285}}{2}$,

the corresponding value of $y = 3, \dfrac{3+\sqrt{285}}{6}, -\dfrac{23}{4}, \dfrac{3-\sqrt{285}}{6}$.

If we draw the graph of these two equations, we shall find that the first equation represents two intersecting straight lines, and the second one a parabola.



For (1) reduces to two linear equations, $x = 2y$ and $x = -3y$. These have for their graphs the two straight lines.

In (2),

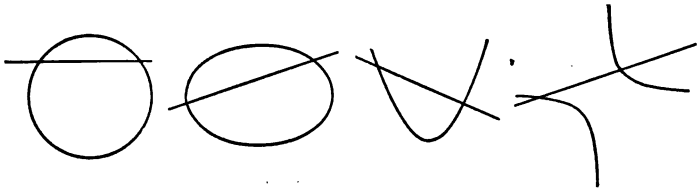
If $x =$	-12	-9	-4	-2	0	3	6	8	9
Then $y =$	-9	8	23	$24\frac{1}{3}$	23	16	3	-9	-16

Plotting these points and drawing the curve, letting each square represent two units, we have the parabola. The four pairs of roots given above are represented on the figure by I, II, III, IV, in order.

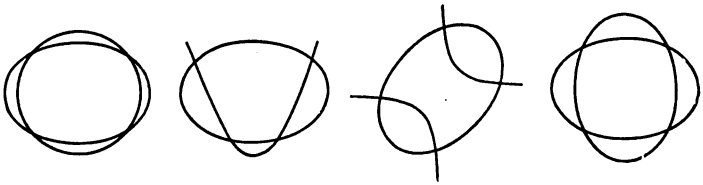
Exercise 128. Simultaneous Quadratics

Examples 1 to 10, oral — Examples 11 to 18, written

1. In how many points can a straight line intersect a circle? intersect an ellipse? a parabola? an hyperbola?



2. How many roots in the system $x + 2y = 7$, $x^2 + y^2 = 16$?
3. How many roots in the system $x - 4y = 5$, $x^2 + 3y^2 = 9$?
4. How many roots in the system $x = 2y$, $y^2 = 4x$?
5. How many roots in the system $3x + y = 9$, $xy = 6$?
6. In how many points can an ellipse intersect a circle? intersect a parabola? an hyperbola? another ellipse?



7. How many roots in the system $x^2 + 5y^2 = 16$, $x^2 + y^2 = 9$?
8. How many roots in the system $x^2 + 5y^2 = 16$, $y^2 = x + 3$?
9. How many roots in the system $x^2 + 5y^2 = 16$, $xy = 1$?
10. How many roots in the system $5x^2 + y^2 = 16$, $x^2 + 5y^2 = 16$?

Solve and plot the equations in the following examples:

11. Ex. 2.	13. Ex. 4.	15. Ex. 7.	17. Ex. 9.
12. Ex. 3.	14. Ex. 5.	16. Ex. 8.	18. Ex. 10.

224. Two Equations Homogeneous except for Absolute Terms. If two quadratic equations are homogeneous except for the absolute terms, they can also be solved. For if we eliminate the absolute term from either equation, we have a case exactly like the one discussed on page 231.

Solve the equations

$$x^2 + xy + 2y^2 = 44 \quad (1)$$

$$2x^2 - xy + y^2 = 16 \quad (2)$$

Multiplying (1) by 4, $4x^2 + 4xy + 8y^2 = 176.$ (3)

Multiplying (2) by 11, $22x^2 - 11xy + 11y^2 = 176.$ (4)

Dividing (4) − (3) by 3, $6x^2 - 5xy + y^2 = 0.$

Factoring, $(3x - y)(2x - y) = 0.$

From $3x - y = 0$ we have $y = 3x.$

From $2x - y = 0$ we have $y = 2x.$

Substituting $3x$ for y in (2),

$2x^2 - 3x^2 + 9x^2 = 16.$

$8x^2 = 16.$

$x^2 = 2.$

$x = \pm\sqrt{2}.$

$y = 3x = \pm 3\sqrt{2}.$

Substituting $2x$ for y in (2),

$2x^2 - 2x^2 + 4x^2 = 16.$

$4x^2 = 16.$

$x^2 = 4.$

$x = \pm 2.$

$y = 2x = \pm 4.$

Therefore, when $x = +\sqrt{2}, -\sqrt{2}, +2, -2,$

the corresponding value of $y = +3\sqrt{2}, -3\sqrt{2}, +4, -4.$

Exercise 129. Simultaneous Quadratics

All work written

Solve the following equations:

1. $x^2 - xy + y^2 = 21$
   $x^2 + 3xy - 2y^2 = 38$

2. $x^2 - xy + y^2 = 21$
   $y^2 - 2xy = -15$

3. $x^2 + 2xy - y^2 = 50$
   $3x^2 - xy + y^2 = 75$

4. $x^2 + 7xy + y^2 = 45$
   $5x^2 - xy + 2y^2 = 33$

5. $2x^2 - xy + 2y^2 = 12$
   $3x^2 - 2xy + 4y^2 = 20$

6. $3x^2 + xy + 5y^2 = 81$
   $2x^2 - 3xy + 4y^2 = 27$

Exercise 130. Simultaneous Quadratics

Examples 1 to 4, oral — Examples 5 to 20, written

1. In the equations $x + y = 9$ and $x^2 + y^2 = 41$, if we find that $x = 5$ or 4, then, since we can interchange x and y, what do we know about the values of y?

2. In the equations $xy = 15$, $x + y = 8$, if we find that $x = 3$ or 5, what is the rest of the solution?

3. In the equations $\frac{1}{x} + \frac{1}{y} = \frac{5}{12}$, $\frac{1}{xy} = \frac{1}{24}$, if we find that $x = 4$ when $y = 6$, what are the other two values of x and y? Why?

4. In the equations $x^3 - y^3 = 98$, $x - y = 2$, can you divide the first by the second? Why is this better than substituting the value of y (in the form $y = x - 2$) in the first equation?

Solve and plot the following equations:

5. $x^2 - y = 3$
$x - y = -3$

6. $3y = 2x - 1$
$9y = x^2 + 2$

7. $3x - y = 5$
$3x + y^2 = 17$

8. $5x - 3y + 6 = 0$
$5x - 2y^2 + 5 = 0$

9. $3x + y - 12 = 0$
$4x^2 - 5x - y^2 = 12$

10. $4x - 13y + 11 = 0$
$x^2 - 2xy - 2y^2 = 22$

11. $2x - y - 7 = 0$
$x^2 - 7x - 2y^2 = -17$

12. $3x + y - 16 = 0$
$x^2 + 4y^2 = 29$

13. $x + y = 12$
$xy = 35$

14. $x - y = 22$
$xy = 663$

15. $x + y = 38$
$xy = 357$

16. $x^2 + y^2 = 661$
$x^2 - y^2 = 589$

17. $3x^2 - y^2 = 59$
$2x^2 + 3y^2 = 98$

18. $5x^2 + 2y^2 = 220$
$2x + 5y = 54$

19. $16x^2 - 9y^2 = 0$
$2xy - 5x + 6y = 33$

20. $5x^2 + y^2 = 126$
$5y^2 + x^2 = 30$

Exercise 131. Problems involving Simultaneous Quadratics

All problems for written work. Reject meaningless results

1. The sum of two numbers is 32 and their product is 255. Find the numbers.

2. The difference of two numbers is 4 and their product is 437. Find the numbers.

3. The sum of two numbers is 24 and the sum of their reciprocals is $\frac{3}{16}$. Find the numbers.

4. The sum of two numbers is 74 and the sum of their square roots is 12. Find the numbers.

5. The sum of two numbers is 23 and the sum of their cubes is 3059. Find the numbers.

6. The difference of the cubes of two numbers is 218 and the difference of the numbers is 2. Find the numbers.

7. The sum of the squares of two numbers is 89 and the product of the numbers is 40. Find the numbers.

8. The sum of two numbers added to the sum of their squares is 686, and the difference of the numbers added to the difference of their squares is 74. Find the numbers.

9. The product of two numbers is 91 greater than ten times the first number, and 51 greater than ten times the second number. Find the numbers.

10. There are two numbers formed by the same two digits in reverse order. The sum of the numbers is 55 times the difference between the two digits, and the difference between the squares of the two numbers is 1980. Find the numbers.

Let $10x + y$ and $10y + x$ represent the numbers.

11. A strip of cloth when wet shrinks $12\frac{1}{2}\%$ in length and $6\frac{1}{4}\%$ in width. If the number of square yards is diminished $11\frac{1}{2}$ when it is wet, and if the sum of the length and width is $8\frac{1}{16}$ yd. less than before, what were the dimensions of the strip before it was wet?

12. A workman wishes to enlarge a drawing of a rectangle so that the area will be twice the original area, keeping the ratio of the length to the width unchanged. The original drawing is 8 in. by 10 in. Find the sides of the enlarged rectangle.

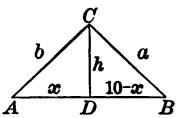
13. A rectangular field is 30 yd. wide, and the length exceeds the width by $66\frac{2}{3}\%$. How much must the width be decreased and the length increased so that the area will remain the same while the perimeter is increased 30 yd.?

14. Two points move, each at a uniform rate, on the sides of the right angle of a right triangle ABC, away from the vertex A, starting from the two points P and Q, 3 in. and 4 in. respectively from the vertex A. After 2 sec. they are 10 in. apart, and after 6 sec. they are 20 in. apart. Find the rate of each.

15. There are two lines such that if they are made the sides of the right angle of a right triangle, the hypotenuse will be 35 in.; but if one is made the hypotenuse and the other is made a side, the square on the other side will be 343. Find their lengths.

16. A boat's crew, rowing at half their usual rate, row 2 mi. down a river and back in 1 hr. 40 min. At their usual rate in still water they would have gone over the same course in 40 min. Find their rate of rowing in still water and the rate of the current.

17. In the figure below, $c = AB = 10$, $b = 8$, $a = 7$. Required to find the length of the perpendicular CD.



We have $x^2 + h^2 = b^2$, and $\overline{DB}^2 + h^2 = a^2$. But $\overline{DB} = 10 - x$. Therefore $x^2 + h^2 = 64$, and $(10 - x)^2 + h^2 = 49$. Hence we have two quadratics.

18. In Ex. 17, suppose $c = 15$, $b = 10$, and $a = 9$. Find the length of CD, and then find the area of the triangle.

19. Two cubes have together the volume 407 cu. in., and the sum of one edge of the one and one edge of the other is 11 in. Find the volume of each.

CHAPTER XVIII

RATIO AND PROPORTION

225. Ratio. The relation of one number to another number of the same kind, as indicated by division, is called the *ratio* of the first to the second.

The ratio of $a+b$ to $c+d$ is indicated thus: $\frac{a+b}{c+d}$, or $a+b:c+d$.

Hence all ratios may be looked upon as fractions.

226. Terms of a Ratio. The two numbers involved in a ratio are called the *terms* of the ratio. The dividend is called the *antecedent* and the divisor is called the *consequent.*

Thus we have

$$\frac{a}{b}=\frac{\text{antecedent}}{\text{consequent}}=\frac{\text{numerator}}{\text{denominator}}=\frac{\text{dividend}}{\text{divisor}}.$$

If $\frac{a}{b}=\frac{2}{3}$, we say that a is to b as 2 is to 3.

227. Greater and Less Inequality. If the absolute value of a ratio is greater than 1, the ratio is called a *ratio of greater inequality.* If the absolute value of a ratio is less than 1, the ratio is called a *ratio of less inequality.*

Thus $\frac{3}{2}$ is a ratio of greater inequality, and $\frac{2}{3}$ is a ratio of less inequality. The symbol $>$ is read "is greater than," and the symbol $<$ is read "is less than."

228. Laws of Ratio. The following are the important laws of a ratio whose terms are positive.

1. *Both terms of a ratio may be multiplied, or both may be divided, by the same number without changing the value of the ratio.*

A ratio being a fraction, the laws of fractions are true for ratios.

2. *A ratio of greater inequality is diminished, and a ratio of less inequality is increased, by adding the same positive number to both terms.*

Let a be the antecedent and b the consequent.

Then $\frac{a+n}{b+n} >$ or $< \frac{a}{b}$,

according as $ab + bn >$ or $< ab + an$,

or as $bn >$ or $< an$,

or as $b >$ or $< a$.

That is, if $a > b$, then $\frac{a+n}{b+n} < \frac{a}{b}$,

and if $a < b$, then $\frac{a+n}{b+n} > \frac{a}{b}$.

3. *A ratio of greater inequality is increased, and a ratio of less inequality is decreased, by subtracting the same positive number from both terms.*

We must first notice that

$$-4 > -5, \text{ but } 4 < 5,$$

and that, in general, if $-b > -a$, then $b < a$.

That is, changing the sign changes the order of inequality.

We see that $\frac{a-n}{b-n} >$ or $< \frac{a}{b}$,

according as $ab - bn >$ or $< ab - an$,

or as $-bn >$ or $< -an$,

or as $b <$ or $> a$.

4. *In a series of equal ratios the sum of all the antecedents is to the sum of all the consequents as any antecedent is to its consequent.*

Consider the three ratios $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$, and let each equal r.

Since $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = r$, $\therefore a = br$, $c = dr$, and $e = fr$.

Adding, $a + c + e = (b + d + f)r$.

Dividing, $\frac{a+c+e}{b+d+f} = r = \frac{a}{b} = \frac{c}{d} = \frac{e}{f}$.

Exercise 132. Ratios

Examples 1 to 4, oral — Examples 5 to 17, written

1. Express in simplest fractional form the ratio of 10 to 20; of 32 to 40; of 45 to 54; of 49 to 56; of 26 to 39.

2. Express in simplest form, as improper fractions, the ratio of 15 to 10; of 20 to 12; of 35 to 21; of 81 to 63.

3. How is the ratio of 9 to 5 changed by multiplying both terms by 2? by adding 2 to both terms? by subtracting 2 from both terms?

4. How is the ratio of 6 to 9 changed by multiplying both terms by 3? by dividing both terms by 3? by adding 3 to both terms? by subtracting 3 from both terms?

Simplify the following ratios:

5. $a^2 - b^2 : a^3 - b^3$.

6. $\dfrac{1}{x+y} : \dfrac{1}{x^2+2xy+y^2}$.

7. $a^3 - 1 : a^2 + a + 1$.

8. $1 + \dfrac{a}{b} : 1 - \dfrac{a}{b}$.

9. Separate 50 into two parts having the ratio 2 : 3.

Let x = the smaller part.

Then $50 - x$ = the larger part.

Then $\dfrac{x}{50-x} = \dfrac{2}{3}$.

Therefore $x = 20$, and $50 - x = 30$.

Separate into two parts having the ratios specified:

10. 24, 1 : 2.

11. 35, 2 : 3.

12. 91, 2 : 5.

13. 104, 3 : 5.

14. 272, 7 : 9.

15. 306, 7 : 11.

16. A wheel 30 in. in diameter, making 300 revolutions per minute, is belted to another wheel 15 in. in diameter. Find the speed of the smaller wheel.

17. Two cogwheels are geared together. The distance between their centers is 20 in. What are their diameters if their speeds have a ratio of 4 to 5?

229. Proportion. An expressed equality of two ratios is called a *proportion.*

Thus if $\frac{a}{b}=\frac{c}{d}$, then a, b, c, d, are said to be in proportion or to form a proportion.

This proportion is often read, "a is to b as c is to d," and may be written thus: $a:b=c:d$. In older works it is often written $a:b::c:d$.

230. Terms of a Proportion. In a proportion the first and fourth terms are called the *extremes,* and the second and third terms the *means.*

In the proportion $a:b=c:d$ the extremes are a and d, and the means are b and c.

In the same proportion d is called the *fourth proportional* to a, b, and c.

231. Mean Proportional. If the means of a proportion are the same, this term is called the *mean proportional* between the other two.

Thus m is the mean proportional between x and y if $x:m=m:y$. In this proportion y is called the *third proportional* to x and m.

232. Laws of Proportion. The following are the important laws of proportion:

1. *If four numbers are in proportion, the product of the extremes is equal to the product of the means.*

Let the proportion be $\frac{a}{b}=\frac{c}{d}$.

Multiplying by bd, $ad=bc$.

2. *If the product of two numbers equals the product of two other numbers, either two may be made the means of a proportion, and the other two the extremes.*

Let $ad=bc$.

Dividing by bd, $\frac{ad}{bd}=\frac{bc}{bd}$,

or $\frac{a}{b}=\frac{c}{d}$.

3. *If* $a : b = c : d$, *then* $b : a = d : c$.

In this case the first proportion is said to be taken by *inversion*.

From Law 1, $bc = ad$.

Dividing by ac, $\frac{bc}{ac} = \frac{ad}{ac}$,

or $\frac{b}{a} = \frac{d}{c}$

4. *If* $a : b = c : d$, *then* $a : c = b : d$.

In this case the first proportion is said to be taken by *alternation*.

From Law 1, $ad = bc$.

Dividing by cd, $\frac{ad}{cd} = \frac{bc}{cd}$,

or $\frac{a}{c} = \frac{b}{d}$.

5. *If* $a : b = c : d$, *then* $a + b : b = c + d : d$.

In this case the first proportion is said to be taken by *composition*.

Given $\frac{a}{b} = \frac{c}{d}$.

Adding 1, $\frac{a}{b} + 1 = \frac{c}{d} + 1$,

or $\frac{a+b}{b} = \frac{c+d}{d}$.

6. *If* $a : b = c : d$, *then* $a - b : b = c - d : d$.

In this case the first proportion is said to be taken by *division*.

Given $\frac{a}{b} = \frac{c}{d}$.

Subtracting 1, $\frac{a}{b} - 1 = \frac{c}{d} - 1$,

or $\frac{a-b}{b} = \frac{c-d}{d}$.

7. *The product of the means divided by either extreme equals the other extreme. The product of the extremes divided by either mean equals the other mean.*

The proof is left for the student.

Exercise 133. Proportion

Examples 1 to 6, oral — Examples 7 to 38, written

1. Solve the equation $\frac{1}{2}x = \frac{2}{3}$.

2. In the proportion $\frac{x}{2} = \frac{2}{3}$, find the value of x.

3. In the proportion $x : 2 = 2 : 3$, find the value of x.

Find the value of x in the following proportions:

4. $\frac{x}{4} = \frac{3}{8}$.

5. $\frac{x}{5} = \frac{7}{15}$.

6. $\frac{x}{7} = \frac{11}{21}$.

7. $\frac{2}{3} = \frac{x}{9}$.

8. $\frac{3}{5} = \frac{12}{x}$.

9. $\frac{5}{x} = \frac{25}{35}$.

10. $\frac{7}{x} = \frac{42}{48}$.

11. $\frac{36}{48} = \frac{3}{x}$.

12. $\frac{65}{39} = \frac{5}{x}$.

13. If the means are 3 and 12, and one extreme is 4, what is the other extreme?

14. If $(x - 5) : 5 = 28 : 10$, by what law do we know that $x : 5 = 38 : 10$? What is the value of x?

Find the value of x in the following proportions:

15. $x : 2.4 = 1.21 : 1.1$.

16. $(x - 12) : 12 = 5 : 1$.

17. $2.6 : x = 1.1 : 7.7$.

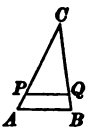
18. $(17 - x) : x = 16 : 18$.

19. If each mean of a proportion is 8, and one extreme is 32, what is the other extreme?

20. In a lever PW, if sufficient power (p) is applied at P, a weight (w) at W can be lifted. If F is the fulcrum, it is known that $p : w$ equals the ratio $FW : FP$. If $FW = 10$ in., $FP = 30$ in., and $w = 240$ lb., what does p equal?

21. If two boys weigh respectively 100 lb. and 120 lb., where must the fulcrum be placed under a 10-foot board so that the boys sitting at the ends will just balance? (Use the law given in Ex. 20.)

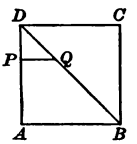
22. It is proved in geometry that a line parallel to the base of a triangle divides the other two sides proportionally. In this figure, if $AP = 1\frac{1}{2}$, $PC = 6$, and $QC = 5$, find the length of BQ.



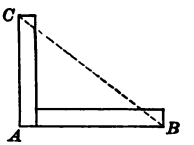
23. If a line parallel to the base of a triangle divides one side into the parts $7\frac{1}{2}$ and $4\frac{1}{2}$, and the other side into the corresponding parts 6 and x, what is the value of x?

24. If two sides of a triangle are 7 and 9, and a line parallel to the base cuts the former into the parts 3 and 4, into what lengths will it cut the latter?

25. In the square here shown, PQ is $\parallel$ to AB. If a side of the square is 10 in., $DB = 14.14$ in. If $DP = 3$ in., what is the length of DQ?

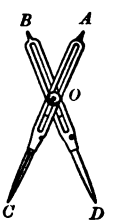


26. Two pieces of timber 1 ft. wide are fitted together at right angles as here shown. AB is 8 ft. long, AC is 6 ft. long, and the distance BC along the dotted line is 10 ft. A carpenter finds it necessary to saw along the dotted line. Find the length of the slanting cut across the upright piece; across the horizontal piece. $x : 10 = 1 : 6$ is one proportion.

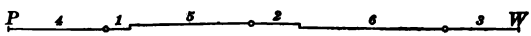


27. It is proved in geometry that two triangles of the same shape have their corresponding sides proportional. Such triangles are said to be *similar*. In Ex. 22 the triangles ABC and PQC are similar. If $AB = 8$, $AC = 7\frac{1}{2}$, and $PC = 6$, what is the length of PQ?

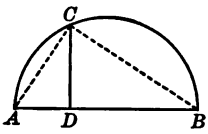
28. The figure at the right represents a pair of proportional compasses used by draftsmen. By adjusting the screw at O, the lengths OA and OC, and the corresponding lengths OB and OD, may be varied proportionally. The triangle formed by O, A, and B is always similar to the triangle formed by O, C, and D. If $OA = 3$ in. and $OC = 5$ in., then AB is what part of CD?



29. In the following combination of levers a power of 100 lb. is applied at P. What weight can be raised at W?

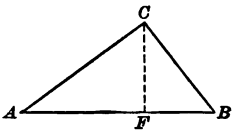


30. It is proved in geometry that if AB is the diameter of the semicircle here shown, CD is a mean proportional between AD and DB. If $AD = 3$ and $CD = 4$, what is the length of DB?



31. In the same figure, if $DB = 4\,AD$ and $CD = 12$, what are the lengths of AD and DB?

32. If a perpendicular is let fall from the vertex of the right angle upon the hypotenuse of a right triangle, it divides the right triangle into two triangles similar to the original triangle and to each other. In the given figure, if $AF = 5$ and $CF = 3.5$, what is the length of FB?



33. In the same figure, if $AB = 7.45$ and $AC = 6.1$, what is the length of AF?

34. If a boy $4\frac{3}{4}$ ft. tall casts a shadow $4\frac{1}{2}$ ft. long at the same time that the school building casts a shadow $67\frac{1}{2}$ ft. long, how high is the school building?

35. If a boy, lying down with his eye to the ground, sights over the top of a 10-foot pole held vertically $6\frac{1}{4}$ ft. from his eye, and can just see the top of a tree $37\frac{1}{2}$ ft. from his eye, how tall is the tree?

36. Show that if $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$. This is called the taking of the original proportion by *composition and division.*

37. Solve the equation $\frac{x+2}{x-2} = \frac{10}{7}$ by the principle of Ex. 36.

38. Solve the equation $\frac{x-7}{x+7} = \frac{2}{3}$ by the principle of Ex. 36.

APPENDIX

233. Cumulative Review. The following collection of exercises is arranged on a cumulative plan that permits the student, on the completion of each chapter, to review not only the subject just studied, but all of the work that has been covered up to that point. It may be used as circumstances seem to warrant.

Exercise 134. Review of Chapter I

1. Define formula, monomial, polynomial, binomial, trinomial, terms of a polynomial, symbols of aggregation, equation, members of an equation, axiom.

2. Give four illustrations of a formula.

3. State the order of operations in an algebraic expression.

4. What is meant by evaluating an expression? Illustrate.

5. Given the formulas $c = \pi d$ and $c = 2\pi r$, derive from them formulas for d and r.

6. If r is the rate for 1 yr., what is the interest on p dollars for 3 mo.? for 6 mo.? for 15 da.? for n years?

7. Given the formula $A = \pi R^2 - \pi r^2$, find the value of A when $R = 14$ and $r = 7$, taking $3\frac{1}{7}$ for the value of π.

Solve the following equations:

8. $2.45 + x = 7.12$.

9. $x - 3.26 = 2.98$.

10. $3x - 1.9 = 3.8$.

11. $7.62 + x = 12$.

12. $2.34x + 2 = 9.02$.

13. $3.19x - 4 = 11.95$.

14. $\frac{2}{3}x = 18$.

15. $\frac{4}{5}x = 3.5$.

16. $\frac{3}{16}x = 7.2$.

17. $1\frac{5}{12}x = 0.75$.

18. $\frac{7}{8}x = 0.49$.

19. $1\frac{1}{2}x = 1.21$.

20. $x \div 1.9 = 2.4$.

21. $3x \div 7 = 8.1$.

22. $7x \div 3 = 4.9$.

23. $\frac{1}{2}x \div 7 = 9.3$.

24. $2x \div 9 = 8.4$.

25. $2\frac{1}{2}x \div 2 = 12$

Exercise 135. Review of Chapters I and II

1. An automobile traveling at the rate of d miles an hour goes $5\frac{1}{2}$ mi. in 11 min. Find the value of d.

2. A horse is tethered by a rope 42 ft. long. Over how many square feet can he graze? (Take $3\frac{1}{7}$ for π.)

3. An iron pillar having a circular cross section is 8.4 in. in diameter. What is the area of the cross section?

4. Given the formula $V = e^3$, find the value of V when $e = 7$ Find the value of e when $V = 27$.

5. Solve the equation $17 - x = 14x - 58$.

6. Rule some paper and draw a line showing the growth of the sales of a manufacturer, the sales being as follows: 1910, \$100,000; 1911, \$140,000; 1912, \$180,000; 1913, \$230,000; 1914, \$290,000; 1915, \$370,000; 1916, \$480,000.

7. Give four illustrations of a negative number, taken from your experience.

8. Add the following: 7.2, 8.9, −3.4, 6.2, −8.1, 5.3, −13.2.

Subtract as indicated:

9. $3.4 - 1.9$.	12. $5.1 - (-3.1)$.	15. $-5.3 - 9.8$.
10. $1.9 - 3.4$.	13. $2.7 - (-4.9)$.	16. $-5.3 - (-9.8)$.
11. $2.6 - 5.8$.	14. $8.6 - (-9.7)$.	17. $-9.8 - (-5.3)$.

Multiply as indicated:

18. -17×6.3.	20. $42 \times (-37)$.	22. $-7 \times (-9.3)$.
19. -9.2×46.	21. $5.6 \times (-4.8)$.	23. $-9 \times (-8.7)$.

Divide as indicated:

24. $-12.5 \div 2.5$.	26. $12.5 \div (-2.5)$.	28. $-44.4 \div (-3.7)$.
25. $-8.68 \div 7$.	27. $8.68 \div (-14)$.	29. $-58.8 \div (-4.9)$.

30. How much difference in price is there in selling an article at \$17.75 below cost and at \$24.50 above cost?

Exercise 136. Review of Chapters I–III

1. Define and illustrate factor, literal factor, and numerical factor.

2. Distinguish between coefficient and exponent, and give two illustrations of each.

3. Define and illustrate power and root.

4. Define and illustrate absolute term, similar terms.

5. The terms of a polynomial are $7\,a^3$, $-\,9$, $4\,a$, and $-\,2\,a^2$. Write the polynomial.

6. The factors of a monomial are $7\,a^3$, $-\,9$, $4\,a$, and $-\,2\,a^2$. Write the monomial.

7. If $f(d)$ is πd, what is $f(7)$? $f(14)$? (Take $3\frac{1}{7}$ for π.)

8. Find the fourth root of 256.

9. Write three functions of r that you have studied.

10. If a man is \$375 in debt, how will you represent his capital algebraically? How will you represent the capital of a man who is half as much in debt?

Add as indicated:

11. $9.8 + (-7.2)$.
12. $0.9 + (-8.8)$.
13. $-9.7 + 16.9$.
14. $-8.6 + 21.4$.
15. $-7.7 + (-8.9)$.
16. $-5.9 + (-9.5)$.

Subtract as indicated:

17. $16.3 - 7.9$.
18. $16.3 - (-7.9)$.
19. $-16.3 - 7.9$.
20. $-6.3 - (-9)$.
21. $-3.1 - (-9.6)$.
22. $-4.2 - (-8.7)$.

Multiply as indicated:

23. $48 \times (-7)$.
24. $(-7) \times (-48)$.
25. $7 \times (-5.9)$.
26. -3.8×6.
27. $-1.9 \times (-4.2)$.
28. $-3\frac{2}{3} \times (-2\frac{1}{3})$.

Divide as indicated:

29. $333 \div (-37)$.
30. $-33.3 \div 3.7$.
31. $-33 \div (-7)$.
32. $-3.3 \div (-9)$.
33. $-\frac{2}{3} \div (-\frac{3}{4})$.
34. $-\frac{3}{4} \div (-\frac{2}{3})$.

Exercise 137. Review of Chapters I–IV

1. Evaluate $3a^2 + 4ab - 7b^2$ for $a = 7$, $b = -2$.

2. Evaluate $4x^3 - 3x^2 + 7x - 7$ for $x = 3$; for $x = -5$.

3. Add 2.8, -3.2, 4.9, 7.6, -42.7, and 29.4.

4. From the sum of 19.7 and -12.9 take -9.8.

5. Multiply $27.9 + (-13.8)$ by -3; by $-3.1 + 4.1$.

6. Divide $-24.7 + (-28.5)$ by -2; by -4.

7. Given the formula $a = \frac{1}{2}cr$, find the formula for c in terms of a and r.

8. Name the numerical coefficient in $41a^4b^m$; the numerical exponent; the literal exponent. Evaluate the expression for $a = 2$, $b = 5$, $m = 2$.

9. From the sum of 43, -87, 29, and -12, subtract the sum of -19, 4.1, 6.9, -2, and -7.

10. Multiply the sum of 21, -32, 48, and -56, by the sum of -3, 4, and -8.

Add the following expressions:

11.

$$\begin{array}{r} -7a^2 + 3a - 6 \\ 2a^2 - 6a + 9 \\ \underline{8a^2 - 9a + 7} \end{array}$$

12.

$$\begin{array}{r} 4x^2 - 3x + 19 \\ -2x^2 - 7x - 29 \\ \underline{9x^2 + 8x - 32} \end{array}$$

13.

$$\begin{array}{r} 5p^3 + 6p^2 - 7p + 8 \\ 8p^3 - 7p^2 - 9p + 5 \\ 7p^3 - 5p^2 + 8p - 7 \\ \underline{9p^3 + 4p^2 - 8p + 2} \end{array}$$

14.

$$\begin{array}{r} x^4 - 3x^3 + 4x^2 - 7x + 2 \\ 5x^4 - 7x^3 - 3x^2 + 8x - 5 \\ \underline{9x^4 + 8x^3 - 7x^2 + 9x + 7} \end{array}$$

15.

$$\begin{array}{r} x^4 + 7x^2 - 9 \\ 52x^3 + 3x^2 - 81x + 17 \\ \underline{24x^4 - 27x^2 + 96x - 13} \end{array}$$

16.

$$\begin{array}{r} a^4 - a^3b + a^2b^2 - ab^3 + b^4 \\ a^4 + 2a^3b - 3a^2b^2 + 7ab^3 - b^4 \\ a^4 - 3a^3b - 7a^2b^2 + 8ab^3 - b^4 \\ \underline{a^4 + 2a^3b + 9a^2b^2 - 14ab^3 + b^4} \end{array}$$

Exercise 138. Review of Chapters I–V

1. What is the value of $261 \div 9 + 2 \times 348 \div 4 - 2$?

2. How do you add monomials? Illustrate.

3. Add $x^4 - 3x^2 + 2x - 1$ and $17x^3 - 23x^2 - 48x + 36$.

4. Solve the equation $17x - 23x + 49 = 84 - 11x$.

5. How do you subtract a negative quantity from a positive quantity? Illustrate by the thermometer.

6. How do you subtract a positive quantity from a negative quantity? Illustrate.

7. How do you subtract a negative quantity from a negative quantity? Illustrate.

8. Given the formula $a = \frac{1}{2}bh$, find the formula for h.

Add as indicated:

9. $7\frac{1}{3} + (-5\frac{1}{4})$.
10. $8\frac{3}{4} + (-2\frac{1}{2})$.
11. $-4\frac{7}{8} + 5\frac{1}{3}$.
12. $-6\frac{3}{8} + 9\frac{3}{4}$.
13. $-9\frac{3}{5} + (-2\frac{1}{3})$.
14. $-6\frac{3}{8} + (-3\frac{3}{4})$.

Subtract as indicated:

15. $12\frac{1}{3} - 6\frac{1}{2}$.
16. $12\frac{1}{3} - (-6\frac{1}{2})$.
17. $-7\frac{1}{2} - 2\frac{3}{8}$.
18. $-9\frac{1}{4} - (-1\frac{7}{8})$.
19. $-5\frac{3}{4} - (-6\frac{1}{3})$.
20. $-2\frac{1}{3} - (-4\frac{1}{4})$.

Multiply as indicated:

21. $2\frac{1}{2} \times (-3\frac{1}{4})$.
22. $-2\frac{1}{4} \times 3\frac{1}{2}$.
23. $5 \times (-3\frac{7}{8})$.
24. $-4\frac{2}{3} \times 7$.
25. $-\frac{4}{5} \times (-\frac{3}{8})$.
26. $-\frac{7}{8} \times (-\frac{2}{3})$.

Divide as indicated:

27. $11.9 \div (-7)$.
28. $-1.19 \div 0.7$.
29. $-\frac{3}{8} \div (-\frac{3}{4})$.
30. $-\frac{7}{8} \div (-\frac{5}{16})$.
31. $-23.8 \div (-1.7)$.
32. $-4.76 \div (-1.7)$.

Subtract the following:

33.

$$\begin{array}{r} 21x^3 - 23x^2 + 17x - 13 \\ \underline{18x^3 - \ 4x^2 - \ 9x + 27} \end{array}$$

34.

$$\begin{array}{r} 19a^3 + 23a^2 - 42a - 48 \\ \underline{26a^3 - 14a^2 - 12a + 27} \end{array}$$

Exercise 139. Review of Chapters I–VI

1. What is meant by the degree of a term? by the degree of a polynomial? Illustrate each.

2. What is meant by a homogeneous polynomial? Write a homogeneous polynomial of the fourth degree.

3. Write a monomial of the seventh degree involving three letters; of the eighth degree involving the same three letters. Write the product of these two monomials.

4. The area of a circle is πr^2. What is the area of three circles of radius r? Evaluate the result for $\pi = 3\frac{1}{7}$, $r = \frac{7}{8}$.

5. What is the sum of $-17\,a^2b$, $29\,a^2b$, and $-8\,a^2b$? Evaluate the result for $a = -3$, $b = 4$.

6. On a line mark off $+7$ and -4. Find the difference between these two numbers. Show that it may be either positive or negative.

Add the following:

7.

$$\begin{array}{r} 28\,a^3 - 42\,a^2 + 27\,a - 3 \\ 14\,a^3 + 23\,a^2 - 36\,a + 7 \\ \underline{-\,5\,a^3 - 31\,a^2 + 46\,a + 9} \end{array}$$

8.

$$\begin{array}{r} 31\,x^3 - 33\,x^2 + 31\,x - 27 \\ -\,42\,x^3 + 48\,x^2 - 32\,x + 63 \\ \underline{-\,37\,x^3 - 21\,x^2 - 49\,x - 21} \end{array}$$

Subtract the following:

9.

$$\begin{array}{r} x^3 - 42\,x^2 + 27\,x - 3 \\ \underline{x^3 - 81\,x^2 + 13\,x - 9} \end{array}$$

10.

$$\begin{array}{r} 5\,x^3 - 4\,x^2y + 3\,xy^2 - y^3 \\ \underline{9\,x^3 - 9\,x^2y - 3\,xy^2 + y^3} \end{array}$$

Multiply:

11. $a^3 + 7\,a^2b - 4\,ab^2 - 41\,b^3$ by $a^2 - 2\,a + 3$.

12. $4\,m^4 - 3\,m^2 + 2\,m - 7$ by $m^2 + 3\,m - 7$.

13. $5\,p^4 + 6\,p^3q - 7\,pq^3 + 8\,q^4$ by $p^2 - 2\,p - q$.

14. $8\,a^2b^2 + 7\,a^3b - 3\,ab^3 + a^4 - b^4$ by $3\,a^2 - 2\,ab + 4\,b^2$.

Exercise 140. Review of Chapters I–VII

1. A steel shaft has a diameter of $3\frac{1}{2}$ in. What is the circumference of the shaft? the area of a cross section?

2. Given the formula $PW = P'W'$, find the value of P when $W = 7$, $P' = 28$, and $W' = 60$.

Solve the following equations:

3. $3x + 4 = x + 38$.
4. $4x - 7 = 2x + 37$.
5. $x + 9 = 7x - 27$.
6. $2x - 3 = 11x - 75$.
7. $\frac{2}{3}x - 3 = x - 7$.
8. $\frac{3}{4}x - 9 = x - 36$.
9. $\frac{1}{3}x + 2 = \frac{2}{3}x - 17$.
10. $\frac{1}{3}x + 5 = \frac{1}{2}x - 12$.

Subtract the following, and check the results:

11.

$$\begin{array}{r} 32x^3 - 4x^2 + 17x - 3 \\ -9x^3 - 9x^2 + 12x - 8 \\ \hline \end{array}$$

12.

$$\begin{array}{r} 4x^4 - 3x^3 + 2x - 7 \\ 5x^4 - 7x^3 - 3x - 9 \\ \hline \end{array}$$

Multiply the following, and check the results:

13. $a^3 - 3a^2 + 4a + 9$ by $a^2 + 3a - 2$.
14. $p^3 - 4p^2 + 2p - 8$ by $3 + p^2 - 2p$.
15. $x^4 - x^2y^2 + y^4$ by $x^4 + x^2y^2 + y^4$.
16. $(a + b)^2 + 3(a + b) - 4$ by $(a + b)^2 - 1$.
17. $a^5 + 3a^2 - 2a + 3$ by $a^3 - a + 7$.
18. Multiply the product of $a - 7b$ and $8a - b$ by the product of $a + 2b$ and $a - 5b$.

Divide:

19. $x^2 - 7x + 12$ by $x - 3$; by $x - 4$; by $x + 1$.
20. $x^2 + x - 72$ by $x + 9$; by $x - 8$; by $x - 2$.
21. $2x^3 - x^2 + 3x - 9$ by $2x - 3$; by $2x + 3$.
22. $6x^3 + 14x^2 - 4x + 24$ by $3x^2 + 2x + 1$; by $x + 1$.
23. $7x^3 + 58x - 24x^2 - 21$ by $x^2 - 3x + 7$; by $x - 2$.

Exercise 141. Review of Chapters I–VIII

1. Evaluate the formula $a = \frac{1}{2}h(b + b')$ for $h = 27, b = 2\frac{1}{2}, b' = 3\frac{3}{4}$; for $h = 32, b = 3.5, b' = 4.5$.

Evaluate the following for $a = 3, b = -2, c = 7$:

2. $a^2 - 3b + 9c$.
3. $a^3 - 4b^2 + c^2$.
4. $a^4 - 9b^3 + 2c$.
5. $(a + b)(b + c)$.
6. $(a - 7b)(c - 4b)$.
7. $(a^2 + b^2)(a^2 - b^2)$.
8. $(a - c)(c - 3b)$.
9. $(a^2 - 15b) \div (c + b)$.

Given $A = a^2 - 2a + 3$, $B = a^2 + 7a - 2$, and $C = a^2 + 9a - 3$, find the following:

10. $A + B$.
11. $A - B$.
12. $A + C$.
13. $B - C$.
14. $A + B + C$.
15. $A + B - C$.
16. $A - B + C$.
17. $A - B - C$.
18. AB.
19. AC.
20. BC.
21. ABC.

Given $P = x^2 - 10x + 21$, $Q = x^2 - 12x + 27$, $R = x - 3$, find the following:

22. $P + Q + R$.
23. $P - Q + R$.
24. $Q - P - R$.
25. $Q - 2P + R$.
26. PR.
27. QR.
28. PQ.
29. PQR.
30. $P \div R$.
31. $Q \div R$.
32. $PQ \div R$.
33. $PQ \div R^2$.

Solve the following equations:

34. $9x - 7 = 7x + 35$.
35. $8x - 9 = 4x + 27$.
36. $7x + 8 = 5x + 86$.
37. $2x - 3 = 39 - 5x$.
38. $\frac{3}{5}x = 58$.
39. $\frac{3}{4}x + 9 = 72$.
40. $\frac{4}{5}x - 8 = 86$.
41. $\frac{7}{8}x + 12 = 65$.

42. What number is it that added to 9.07 equals 38.01?

43. If from two thirds of a number we subtract 3 the result is 17. What is the number? Prove it.

Exercise 142. Review of Chapters I–IX

1. What do you mean by a simple equation? Illustrate. What other names are there for a simple equation?

2. What is an identity? Illustrate. Write an equation that is not an identity.

3. When is an equation said to be satisfied? Illustrate.

4. What is meant by solving an equation? How do you determine that a solution is correct?

5. What is meant by the root of an equation? What other meaning has the word "root"?

6. Distinguish between the use of + and − as signs of operation and as signs of quality.

7. What is the difference between the sum of the squares of two quantities and the square of the sum of the quantities?

8. What is the difference between the sum of the squares of two quantities and the square of the difference of the quantities?

9. Show that the difference between the square of the sum of two quantities and the square of the difference of the quantities is four times the product of the quantities.

10. What is the difference between the cube of the sum of two quantities and the cube of the difference of the quantities?

11. What is the quotient of the sum of the fifth powers of two quantities divided by the sum of the quantities?

12. Is the difference of the sixth powers of two quantities always divisible by the difference of the cubes of the quantities? If so, what is the quotient?

13. Is the sum of the sixth powers of two quantities always divisible by the sum of the squares of the quantities? by the sum of the cubes of the quantities? Prove both statements.

14. Is the difference of the eighth powers of two quantities divisible by the difference of the quantities? Prove it.

Multiply:

15. $-x^3+2x^2y-y^3$ by $4x^2+8xy$.

16. a^2+ab+b^2 by $2a^2-3ab-7b^2$.

17. $x^4-3x^3+2x^2-7x+1$ by x^2+4x-3.

18. $5a^4-7a^3b+4a^2b^2-3ab^3+b^4$ by $a^2-3ab+4b^2$

19. $7x^4-2x^2y^2-19xy^3-7y^4$ by $x^3-x^2y+2xy^2-y^3$.

Divide:

20. x^4-81y^4 by $x^3+3x^2y+9xy^2+27y^3$.

21. x^5-y^5 by $x^4+x^3y+x^2y^2+xy^3+y^4$.

22. a^5+32b^5 by $a^4-2a^3b+4a^2b^2-8ab^3+16b^4$.

23. $2a^4+27ab^3-81b^4$ by $2a^3-6a^2b+18ab^2-27b^3$.

24. $x^4+11x^2-12x-5x^3+6$ by $3+x^2-3x$.

Solve the following equations:

25. $5x-(3x-7)=4x-(6x-35)$.

26. $6x-2(9-4x)+3(5x-7)=10x-(4+16x+35)$.

27. $9x-3(5x-6)+30=0$.

28. $x-7(4x-11)=14(x-5)-19(8-x)-61$.

29. $(x+7)(x-3)=(x-5)(x-15)$.

30. To the double of a certain number we add 14 and obtain as a result 154. What is the number?

31. To four times a certain number we add 16 and obtain as a result 188. What is the number?

32. By adding 46 to a certain number we obtain as a result a number three times as large as the original number. Find the original number.

33. One number is three times as large as another. If we take the smaller from 16 and the greater from 30, the remainders are equal. What are the numbers?

34. Divide the number 92 into four parts such that the first exceeds the second by 10, the third by 18, and the fourth by 24.

Exercise 143. Review of Chapters I–X

1. What do you mean by a factor of an algebraic expression? by a monomial factor? by a binomial factor? Illustrate each of these terms.

2. Are the factors of ab the quantities $+a$ and $+b$, or $-a$ and $-b$? Explain your answer.

3. Are the factors of $-x^2+7x-12$ the quantities $x-3$ and $4-x$, or $3-x$ and $x-4$? Verify your answer by multiplying the binomials together.

4. Evaluate the expression $a^3+3a^2b+3ab^2+b^3$ for $a=3$, $b=5$; for $a=4$, $b=-4$.

5. Give illustrations of the following: coefficient, exponent, trinomial, root of an equation, square root of a number, function.

6. Add $7x^5-3x^4+1$, $9x^3-4x^2+7$, $8x^5-4x^4+9$, $-15x^5+7x^4-9x^3+4x-16$.

7. From $x^6-3x^5+7x^4-4x^3+3x^2-9x+36$ subtract $x^6+8x^5-4x^3+5x-96$.

8. Multiply x^4-3x^2+4x-9 by x^3+x^2-3.

9. Divide $x^4-9x^2+x^3-16x-4$ by x^2+4+4x, and check the result.

10. What is the quotient of the sum of the sixth powers of two quantities divided by the sum of their squares?

Solve the following equations:

11. $(x-8)(x+12)=(x+1)(x-6)$.

12. $(x-2)(7-x)+(x-5)(x+3)-2(x-1)+12=0$.

13. $(2x-7)(x+5)=(9-2x)(4-x)+229$.

14. $14-x-5(x-3)(x+2)+(5-x)(4-5x)=45x-76$.

15. $(x+5)^2-(4-x)^2=21x$.

16. $5(x-2)^2+7(x-3)^2=(3x-7)(4x-19)+42$.

17. $(x-3)(x-4)=x(x-1)-30$.

18. The sum of two numbers is 20, and if three times the smaller number is added to five times the larger, the sum is 84. What are the numbers?

19. The sum of the ages of a father and son is 80 yr. If the age of the son were doubled, he would be 10 yr. older than the father. What is the age of each?

20. A man has six sons, each 4 yr. older than the next younger. The eldest is three times as old as the youngest. What is the age of each?

21. If we add \$24 to a certain sum, the amount will be as much above \$80 as the sum is below \$80. What is the sum?

22. The sum of \$500 is divided among A, B, C, and D. A and B together have \$280, A and C \$260, and A and D \$220. How much has each?

23. If A is twice as old as B, and if he was three times as old as B 22 yr. ago, how old is A?

24. A father is 30 yr. old and his son is 6 yr. old. In how many years will the father be twice as old as the son?

25. A sum of money consists of dollars and quarters, and amounts to \$20. There are 50 coins in all. How many are there of each kind?

26. A man paid \$15.25 with quarters and half dollars, giving 51 pieces of money in all. How many of each kind were there?

27. A man bought 30 lb. of sugar of two different kinds, paying \$1.35 for it all. The better kind cost 5¢ a pound and the poorer kind $3\frac{1}{2}$¢ a pound. How many pounds were there of each kind?

28. Two trains start toward each other at the same time from Buffalo and New York, respectively, 450 mi. apart. The one from New York travels at the rate of 50 mi. an hour, and the other 0.8 as fast. How far from New York will they meet?

Factor the following expressions:

29. $5x^3 - 25x^2$.

30. $16x^4 - 25x^2$.

31. $6x^3 + 18x^2 - 12x$.

32. $49a^2 - 21a + 14$.

33. $y^4 - 2y^3 + y^2$.

34. $45a^9b^7 - 360a^7b^9$.

35. $x^2 - ax - bx + ab$.

36. $ab + ay - by - y^2$.

37. $bc + bx - cx - x^2$.

38. $mx + mn + ax + an$.

39. $cdx^2 - cxy + dxy - y^2$.

40. $ax - ay + by - bx$.

41. The factors of a certain trinomial are $3x^2 - 7$ and $4x^2 + 7$. What is the trinomial?

42. The factors of a certain expression are $a - 7b$ and the square of $a - 7b$. What is the expression? What three prime factors has it?

Factor the following expressions:

43. $x^2 + 36 + 12x$.

44. $x^2 + 196 + 28x$.

45. $x(x + 34) + 289$.

46. $a(a - 8) + 16$.

47. $225 - 30a + a^2$.

48. $x^2 + 361 - 38x$.

49. $z^6 - 34z^3 + 289$.

50. $x^2 + 11x + 24$.

51. $x^2 - 7x + 10$.

52. $a^4 - 1$.

53. $a^8 - 1$.

54. $36x^2 - 49y^2$.

55. $a^4 - 25b^2$.

56. $a^4 - 25b^4$.

57. $(a - b)^2 - c^2$.

58. $x^2 - (a - b)^2$.

59. $12x^2 - 5x - 2$.

60. $12x^2 - 7x + 1$.

61. $289x^2y^2z^2 - 102xy^2z^2d + 9y^2z^2d^2$.

62. $361x^2y^2z^2 - 76abcxyz + 4a^2b^2c^2$.

63. $x^2 + y^2 - z^2 - d^2 - 2xy - 2dz$.

64. Show that one factor of $x^4 + x^2y^2 + y^4$ is $x^2 + xy + y^2$, and find the other factor.

65. Show that one factor of $81x^4 - 34x^2y^2 + y^4$ is $9x^2 + 4xy - y^2$, and find the other factor.

Exercise 144. Review of Chapters I–XI

1. Form an equation whose root is 17 and which contains four terms.

2. What is the sum of $x + x + x + \cdots$ written n times? Evaluate the result for $x = 7\frac{1}{2}$, $n = 17$.

3. If the product of two polynomials is $5x^3 - 3x^4 - x + 1$ and one of them is $1 + 3x^2 - 2x$, what is the other?

4. If the sum of two polynomials is $x^4 - 17x^3 + 14x^2$ and one of them is $27x^3 + 19x^2 - 34$, what is the other?

5. Evaluate $\dfrac{e^c - d^c}{e^2 + ed + d^2}$ for $c = 3$, $d = 4$, $e = 5$.

6. From $2x^2 - 2y^2 - z^2$ take $3y^2 + 2x^2 - z^2$, and from the remainder take $3z^2 - 2y^2 - x^2$.

Simplify the following expressions:

7. $2a - [b - (a - 2b)]$.

8. $3a - [b + (2a - b) - (a - b)]$.

9. $2x + (y - 3z) - [(3x - 2y) + z] + 5x - (4y - 3z)$.

10. $a - [2a + (3a - 4a)] - 5a - \{6a - [(7a + 8a) - 9a]\}$

Find the product of:

11. $x - 3$, $x - 1$, $x + 1$, and $x + 3$.

12. $x^2 - x + 1$, $x^2 + x + 1$, and $x^4 - x^2 + 1$.

13. $a^2 + ab + b^2$, $a^2 - ab + b^2$, $a^4 - a^2b^2 + b^4$.

14. $4a^3 - 4a^2b + ab^2$, $4a^2 + 3ab + b^2$, and $2a^2b + b^3$.

Divide:

15. $x^4 - 81y^4$ by $x - 3y$.

16. $x^5 - y^5$ by $x - y$.

17. $x^5 + y^5$ by $x + y$.

18. $a^5 + 32b^5$ by $a + 2b$.

19. $a^6 - b^6$ by $a - b$.

20. $a^6 - b^6$ by $a^2 - b^2$.

21. $x^4 + x^3 - 24x^2 - 35x + 57$ by $x^2 + 2x - 3$.

22. $18x^4 + 82x^2 - 67x + 40 - 45x^3$ by $3x^2 - 4x + 5$.

Solve the following equations:

23. $8(10-x)=5(x+3)$.

24. $2x-3(2x-3)=1-4(x-2)$.

25. $(x-5)(x+6)=(x-1)(x-2)$.

26. $(2x+3)(3x-2)=x^2+x(5x+3)$.

27. $(5x+3)(3x+5)=x(15x+32)+37$.

28. $(9x-7)(3+7x)=3(21x^2-7x+11)-4x$.

29. Find two numbers differing by 8, such that four times the less exceeds twice the greater by 10.

30. Separate 90 into two parts, such that four times one part equals five times the other.

31. A is twice as old as B, and 20 yr. ago he was three times as old. What is B's age?

Write the product of:

32. $1+a+b$ and $1-a-b$.

33. $x+y$, $x-y$, x^2+y^2, and x^4+y^4.

34. a^2+ab+b^2, a^2-ab+b^2, and $a^4+b^4+a^2b^2$. Check, letting $a=1$, $b=1$.

35. Square $x+y+z$. Check, letting $x=1$, $y=2$, $z=3$.

36. Square $a-b+c-d$. Check, letting $a=4$, $b=3$, $c=2$, $d=1$.

Write the product of:

37. $(x+2)(x+3)$.

38. $(x+1)(x+5)$.

39. $(x-3)(x-6)$.

40. $(x-8)(x-1)$.

41. $(x-8)(x+1)$.

42. $(x-2)(x+5)$.

43. $(x-3)(x+7)$.

44. $(x-2)(x-4)$.

45. $(a+1)(a+11)$.

46. $(m-2a)(n+3a)$.

47. $(p-c)(p-d)$.

48. $(w-4m)(w+m)$.

Reduce the following to lowest terms:

49. $\dfrac{x^2-1}{4x(x+1)}$.

50. $\dfrac{x^2-9x+20}{x^2-7x+12}$.

51. $\dfrac{x^2-2x-3}{x^2-10x+21}$.

52. $\dfrac{x^6+2x^3y^3+y^6}{x^6-y^6}$.

53. $\dfrac{a^3+1}{a^3+2a^2+2a+1}$.

54. $\dfrac{a^2-a-20}{a^2+a-12}$.

55. $\dfrac{(a+b)^2}{a^2-ab-2b^2}$.

56. $\dfrac{a^2+2ab+b^2-c^2}{a^2+ab-ac}$.

Change the following to integral or mixed expressions:

57. $\dfrac{x^2-2x+1}{x-1}$.

58. $\dfrac{3x^2+2x+1}{x+4}$.

59. $\dfrac{3x^2+6x+5}{x+4}$.

60. $\dfrac{a^2-ax+x^2}{a+x}$.

61. $\dfrac{2x^2+5}{x-3}$.

62. $\dfrac{2x^2-5x-2}{x-4}$.

63. $\dfrac{a^2+b^2}{a-b}$.

64. $\dfrac{5x^3-x^2+5}{5x^2+4x-1}$.

Change the following to fractional form:

65. $1-\dfrac{x-y}{x+y}$.

66. $1+\dfrac{x-y}{x+y}$.

67. $3x-\dfrac{1+2x^2}{x}$.

68. $a-x+\dfrac{a^2+x^2}{a-x}$.

69. $\dfrac{2x^2}{x+y}-(x+y)$.

70. $a-1+\dfrac{1}{a+1}$.

Add or subtract as indicated:

71. $\dfrac{1}{x-6}+\dfrac{1}{x+5}$.

72. $\dfrac{1}{x-7}-\dfrac{1}{x-3}$.

73. $\dfrac{1}{1-x}-\dfrac{2}{1-x^2}$.

74. $\dfrac{1}{x-y}+\dfrac{x}{(x-y)^2}$.

Exercise 145. Review of Chapters I–XII

1. What is the degree of $27\,x^4y^5z^6$ with respect to z? to y and z? to x? to x and y? to x and z? to x, y, and z?

2. Write a homogeneous polynomial of the third degree.

3. By letting $a=2$, $b=3$, and $c=4$, show that $a+(b+c)=(a+b)+c=a+b+c$.

4. Using the values of Ex. 3, show that $a+b+c=a+c+b=c+b+a$.

5. Add $5x^4+2x^2-7$, $4x^3+x-9$, $1+x-x^2$, $x^5+x^4-x^3-x^2-7$, and $9x^2+9x^3-12x-4x^4+10$, checking the result by letting $x=1$.

6. Simplify $4a-\{3a-[2a-(a-b)]+5b\}$.

7. Multiply x^4+2x^2+4 by x^4-2x^2+4, and check.

8. Divide $x^3-5ax^2-a^2x+14a^3$ by $x^2-3ax-7a^2$.

9. One factor of $x^3+3x^2-13x-15$ is $x+1$. Find two other factors.

Factor the following expressions:

10. x^2+8x+7.

11. $x^2-17x+60$.

12. $x^2+7x-18$.

13. $x^2-2x-24$.

14. $9x^2+30x+25$.

15. $16x^2-56x+49$.

16. x^2+x-72.

17. $x^2-14x-176$.

18. $81a^4-196a^2b^2$.

19. $729a^6-x^6$.

20. $64x^7+xy^6$.

21. $(x^2-y^2)^2-y^4$.

Find the factors that are common to:

22. $12x^2-17x+6$ and $9x^2+6x-8$.

23. x^4-a^4, $x^2+3ax-4a^2$, and $x^2-5ax+4a^2$.

Factor the following expressions:

24. x^2-3x-4, x^2-x-12, and x^2+5x+4.

25. $6x^2-13x+6$, $6x^2+5x-6$, and $9x^2-4$.

Perform the operations indicated:

26. $\dfrac{a}{(a+b)b} - \dfrac{b}{(a-b)a}$.

27. $\dfrac{5}{2x(x-1)} - \dfrac{3}{4x(x-2)}$.

28. $\dfrac{1+x}{1+x+x^2} - \dfrac{1-x}{1-x+x^2}$.

29. $\dfrac{x}{x-y} + \dfrac{x-y}{y-x}$.

30. $\dfrac{8a^2b^3}{45x^2y} \cdot \dfrac{15xy^2}{24a^3b^2}$.

31. $\dfrac{8x^4y}{15ab^3} \div \dfrac{2x^3}{3ab^2}$.

32. $\dfrac{9x^2y^2z}{10a^2b^2c} \cdot \dfrac{-20a^3b^3c}{18xy^3z}$.

33. $\dfrac{a^3-x^3}{a^3+x^3} \cdot \dfrac{(a+x)^2}{(a-x)^2}$.

34. $\dfrac{a^2+b^2}{a^2-b^2} \div \dfrac{a+b}{a-b}$.

35. $\dfrac{x^2+xy}{x-y} \div \dfrac{x^4-y^4}{(x-y)^2}$.

36. $\dfrac{a^2-4a+3}{a^2-5a+4} \cdot \dfrac{a^2-9a+20}{a^2-10a+21} \cdot \dfrac{a^2-7a}{a^2-5a}$.

37. $\dfrac{b^2-7b+6}{b^2+3b-4} \cdot \dfrac{b^2+10b+24}{b^2-14b+48} \div \dfrac{b^2+6b}{b^3-8b^2}$.

Solve the following equations:

38. $(x-3)(x+5) = (x+1)(2x-3) - x^2$.

39. $(x+4)(x-2) = (x+3)(3x+4) - (2x+1)(x-6)$.

40. $(x-3)(2x+5) = x(x+4) + (x+1)(x+3)$.

41. $(x+2)^2 + 3x = (x-2)^2 + 5(16-x)$.

42. The difference between two numbers is 3, and three times the greater number exceeds twice the less by 18. Find the numbers.

Solve the following equations:

43. $5x - \dfrac{x+2}{2} = 71$

44. $x - \dfrac{3-x}{3} = \dfrac{17}{3}$.

45. $\dfrac{13-2x}{4} = x - \dfrac{6x-8}{2}$.

46. $\dfrac{5x}{2} - \dfrac{5x}{4} = \dfrac{9}{4} - \dfrac{3-x}{2}$.

47. $2x - \dfrac{5x-4}{6} = 7 - \dfrac{1-2x}{5}$.

48. $\dfrac{x+2}{2} = \dfrac{14}{9} - \dfrac{3+5x}{4}$.

Solve the following equations:

49. $\frac{5x+3}{8} - \frac{3-4x}{3} + \frac{x}{2} = \frac{31}{2} - \frac{9-5x}{6}$.

50. $\frac{10x+3}{3} - \frac{6x-7}{2} = 10(x-1)$.

51. $\frac{5x-7}{2} - \frac{2x+7}{3} = 3x - 14$.

52. $\frac{9x+20}{36} = 0.25x - \frac{12-4x}{5x-4}$.

53. $\frac{9(2x-3)}{14} + \frac{11x-1}{3x+1} = x + \frac{2x+11}{7}$.

54. $\frac{10x+17}{18} + \frac{12x+2}{16-13x} = \frac{5x-4}{9}$.

55. $\frac{6a+13}{15} - \frac{3a+5}{5a-25} = 0.4a$.

56. Find the number whose third and fourth parts added together make 14.

57. Find the number whose third part exceeds its fourth part by 14.

58. The half, fourth, and fifth parts of a certain number are together equal to 76. Find the number.

59. The sum of two numbers is 5760 and their difference is equal to one third of the greater. Find the numbers.

60. The sum of two numbers is 98 and their difference is 36. Find the numbers.

61. The sum of two numbers is s and their difference is d. Find the numbers.

62. Divide 45 into two such parts that the first part divided by 2 shall be equal to the second part multiplied by 2.

63. Find a number such that the sum of its fifth and seventh parts shall exceed the difference of its fourth and seventh parts by 99.

Exercise 146. Review of Chapters I–XIII

1. Evaluate $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ for $x = 2, y = 3$
2. If $f(x) = x^5 - 5x^4 + 10x^3$, find the value of $f(2)$.
3. Simplify $12x - x - \{7x - [8x - (9x - \overline{3x - 6x})]\}$.

Multiply:

4. $a^2 + ab + b^2$ by $a^2 - ab + b^2$.
5. $a^3 - 3a^2b + 3ab^2 - b^3$ by $a^2 - 2ab + b^2$.
6. $x^2 - xy + y^2 + x + y + 1$ by $x + y - 1$.
7. $4a^7y - 32ay^4 - 8a^5y^2 + 16a^3y^3$ by $a^6y^2 + 4a^2y^4 + 4a^4y^3$.

Find the product of:

8. $a + b$, $a - b$, $a^2 + b^2$, $a^4 + b^4$, and $a^8 + b^8$.
9. $x + a$, $x + 2a$, $x - 3a$, $x - 4a$, and $x + 5a$.
10. $9a^2 + b^2$, $27a^3 - b^3$, $27a^3 + b^3$, and $81a^4 - 9a^2b^2 + b^4$.
11. $a + b - c$, $a + c - b$, $b + c - a$, and $a + b + c$.

Divide:

12. $x^6 - 2x^3 + 1$ by $x^2 - 2x + 1$.
13. $a^4 + 2a^2b^2 + 9b^4$ by $a^2 - 2ab + 3b^2$.
14. $4x^5 - x^3 + 4x$ by $2 + 3x + 2x^2$.
15. $x^4 - 6xy - 9x^2 - y^2$ by $x^2 + y + 3x$.

16. Two casks contain equal quantities of vinegar. From the first cask 34 qt. are drawn, and from the second 20 gal. The quantity now remaining in one cask is twice that remaining in the other. How many quarts did each cask contain at first? How many gallons?

Write the products:

17. $(x - 4y)(x + y)$.
18. $(a - 2b)(a - 5b)$.
19. $(x^2 + 2y^2)(x^2 + y^2)$.
20. $(x + a)(x - b)$.
21. $(ax - 9)(ax + 6)$.
22. $(x^2 - 3xy)(x^2 + xy)$.

Factor the following expressions:

23. $z^4 + 14z^2 + 49$.

24. $a^4 - b^4$.

25. $a^8 - 1$.

26. $y^4 - ay^3 + by^2 + cy$.

27. $x^2 + 3x + 2$.

28. $y^2 - 50yz + 625z^2$.

29. $4x^3y - 12x^2y^2 + 8xy^3$.

30. $cdz^2 - cyz + dyz - y^2$.

31. $(x+1)^2 - (y+1)^2$.

32. $y^2 - 5ay - 50a^2$.

33. $8a^2 + 14ab - 15b^2$.

34. $6a^2 - 19ac + 10c^2$.

35. One factor of $81a^4 - 28a^2b^2 + 16b^4$ is $9a^2 + 10ab + 4b^2$. What is the other factor?

Find the factors that are common to:

36. $6(a-b)^4$, $8(a^2-b^2)^2$, and $10(a^4-b^4)$.

37. $x^2 - y^2$, $(x+y)^2$, and $x^2 + 3xy + 2y^2$.

Factor the following expressions:

38. $x^2 - 9x - 22$ and $x^2 - 13x + 22$.

39. $x^2 - y^2$, $(x+y)^2$, and $(x-y)^2$.

40. $4ab(a^2 - 3ab + 2b^2)$ and $5a^2(a^2 + ab - 6b^2)$.

Reduce the following fractions to lowest terms:

41. $\dfrac{x^4 + x^2 + 1}{x^2 + x + 1}$.

42. $\dfrac{a^2 + 7a + 10}{a^2 + 5a + 6}$.

43. $\dfrac{a^3 - x^3}{a^2 - x^2}$.

44. $\dfrac{6x^2 - 5x - 6}{8x^2 - 2x - 15}$.

Change the following to fractional form:

45. $a + b - \dfrac{a^2 + b^2}{a + b}$.

46. $7a - \dfrac{2 - 3a + 4a^2}{5 - 6a}$.

47. $3x - \dfrac{5ax - 3}{2a}$.

48. $\dfrac{a+b}{a-b} + 1$.

49. $\dfrac{a-b}{a+b} - 1$.

50. $3x - 10 + \dfrac{41}{x+4}$.

Solve the following equations:

51. $\frac{x}{a}+\frac{y}{b}=c$

$\frac{x}{b}+\frac{y}{a}=-c$

52. $abx+cdy=2$

$ax-cy=\frac{d-b}{bd}$

53. $\frac{1}{2}x+\frac{1}{3}y=1.64$

$0.3x-0.04y=0.4224$

54. $\frac{x+y}{9}=0.009$

$\frac{2x-3y}{3}=0.039$

55. $\frac{x-y+1}{x-y-1}=a$

$\frac{x+y+1}{x+y-1}=b$

56. $(a-b)x=(a+b)y$

$ax-by=\frac{a^2+b^2}{2}$

57. $\frac{1}{3}x+\frac{1}{2}y=3.045$

$1000x-y=74.964$

58. $\frac{x+y}{9}=1.33$

$\frac{3x-2y}{3}=8.52$

59. Divide 60 into two parts, such that one part exceeds the other by 24.

60. The sum of two numbers divided by 2 gives as a quotient 24, and their difference divided by 2 gives as a quotient 17. What are the numbers?

61. Three times the greater of two numbers exceeds twice the less by 10, and the sum of twice the greater and three times the less is 24. What are the numbers?

62. A certain fraction equals $\frac{1}{5}$ when the denominator is increased by 4, and equals $\frac{2}{4}\frac{9}{1}$ when the numerator is diminished by 15. What is the fraction?

63. A railway passenger observed that a train moving in the opposite direction passed him in 2 sec., but when moving in the same direction with him it passed him in 30 sec. Compare the rates of the two trains.

64. A certain fraction equals $\frac{2}{3}$ if 7 is added to the numerator, and equals $\frac{3}{8}$ if 7 is subtracted from the denominator. What is the fraction?

Exercise 147. Review of Chapters I–XIV

1. Given the formula $a = \frac{1}{2}h(b + b')$, find the formula for b'. Evaluate the result for $a = 40$, $h = 8$, $b = 6$.

2. Which increases the more rapidly when r increases, the area of a circle or the circumference? Why?

3. Add $4x^6 - 7x^3 + 9x - 19$, $5x^4 - 17x^2 + 3$, $9x^6 - 7x^5 + 4x^4 - 4$, $8x^5 - 5x^4 + 7x^2 - 2$, $12x^3 - 17x + 5$, and $6x^2 - 9x + 7$.

4. From the sum of $x^3 - 9x + 7$ and $x^2 + 7x - 9$ subtract the sum of $x^3 + x^2 - 3$ and $x^2 + 2x - 4$.

5. Multiply $x^3 - x^2 + 3x - 9$ by $x^2 - 7x + 4$ and check the result.

Divide:

6. $x^4 + 64$ by $x^2 + 4x + 8$.

7. $1 - x - 3x^2 - x^5$ by $1 + 2x + x^2$.

8. $x^4 + 9x^2y^2 - 6x^3y - 4y^4$ by $x^2 - 3xy + 2y^2$.

9. $x^5 + x^3 + x^4y + y^3 - 2xy^2 - x^3y^2$ by $x^3 + x - y$.

Write the following expressions in expanded form:

10. $(2x + 1)^3$.

11. $(2a + 5b)^2$.

12. $(3ax - 4x^2)^2$.

13. $(5xy + 2)^2$.

14. $(ab + cd)^2$.

15. $(x - 2y)^3$.

16. $(2x - y)^3$.

17. $(3x + y)^3$.

18. $(x + 3y)^3$.

19. $(2x + 3y)^3$.

Write the quotient of:

20. $\dfrac{b^3 - 125}{b - 5}$.

21. $\dfrac{a^3 - 216}{a - 6}$.

22. $\dfrac{1 + 8a^3}{1 + 2a}$.

23. $\dfrac{8a^3x^3 + 1}{2ax + 1}$.

24. $\dfrac{729a^3 + 216b^3}{9a + 6b}$.

25. $\dfrac{64a^3 + 1000b^3}{4a + 10b}$.

Perform the operations indicated in the following:

26. $\dfrac{x^2+x-2}{x^2-7x} \cdot \dfrac{x^2-13x+42}{x^2+2x}$.

27. $\dfrac{2a(x^2-y^2)^2}{cx} \div \dfrac{(x-y)(x+y)^2}{x^2}$.

28. $\dfrac{x^2-y^2}{x^2-3xy+2y^2} \cdot \dfrac{xy-2y^2}{x^2+xy} \cdot \dfrac{x^2-xy}{(x-y)^2}$.

29. $\dfrac{a^3-3a^2b+3ab^2-b^3}{a^2-b^2} \div \dfrac{2ab-2b^2}{3} \cdot \dfrac{a^2+ab}{a-b}$.

30. $\dfrac{(a+b)^2-c^2}{a^2-(b-c)^2} \div \dfrac{c^2-(a+b)^2}{c^2-(a-b)^2}$.

Solve the following equations:

31. $\dfrac{7x+5}{6} - \dfrac{5x-6}{4} = \dfrac{8-5x}{12}$.

32. $\dfrac{x+4}{3} - \dfrac{x-4}{5} = 2 + \dfrac{3x-1}{15}$.

33. $\dfrac{9x+20}{36} = \dfrac{4(x-3)}{5x-4} + \dfrac{x}{4}$.

34. $\dfrac{9(2x-3)}{14} + \dfrac{11x-1}{3x+1} = \dfrac{9x+11}{7}$.

35. $\dfrac{x-3}{4(x-1)} = \dfrac{x-5}{6(x-1)} + \dfrac{1}{9}$.

36. A certain fraction equals 2 when 7 is added to its numerator, and equals 1 when 1 is subtracted from its denominator. What is the fraction?

Plot the following equations, and solve:

37. $2x+3y=7$
$4x-5y=3$

38. $x-2y=4$
$2x-y=5$

39. $3x-5y=51$
$2x+7y=3$

40. $5x+4y=58$
$3x+7y=67$

Exercise 148. Review of Chapters I–XV

Plot the following equations, and solve:

1. $3x - 4y = -5$
   $4x - 5y = 1$

2. $11x + 3y = 100$
   $4x - 7y = 4$

3. $x + 49y = 50$
   $49x + y = 50$

4. $2x - 7y = 8$
   $4y - 9x = 19$

5. $3x - 4y = 2$
   $7x - 9y = 7$

6. $7x - 5y = 24$
   $4x - 3y = 11$

7. $3x + 2y = 32$
   $20x - 3y = 1$

8. $11x - 7y = 37$
   $8x + 9y = 41$

9. Find the factors of $x^2(x^2 - 1)$ and $2x(x^2 - x - 2)$.

10. Factor $x^2 - 2xy + y^2 - c^2 + 2cd - d^2$.

11. Simplify $\dfrac{3xy - 4}{x^2y^2} - \dfrac{5y^2 + 7}{xy^3} - \dfrac{6x^2 - 11}{x^3y}$.

12. Divide $\dfrac{(a+b)^2 - (c+d)^2}{(a+c)^2 - (b+d)^2}$ by $\dfrac{(a-c)^2 - (d-b)^2}{(a-b)^2 - (d-c)^2}$.

13. Solve the equation $\dfrac{10x + 17}{18} - \dfrac{12x + 2}{13x - 16} = \dfrac{5x - 4}{9}$.

14. A rectangle has its length and breadth respectively 5 ft. longer and 3 ft. shorter than the side of an equivalent square. Find its area.

Solve the following equations:

15. $\dfrac{2}{x+3} = \dfrac{3}{y-2}$
    $\dfrac{x+3}{3} = \dfrac{y-2}{5} + \dfrac{2}{15}$

16. $2x - \dfrac{y-3}{5} = 4$
    $3y + \dfrac{x-2}{3} = 9$

17. $\dfrac{x-4}{5} - \dfrac{y+2}{10} = 0$
    $\dfrac{x}{6} + \dfrac{y-2}{4} = 3$

18. $\dfrac{3x + 12y}{11} = 9$
    $\dfrac{1 - 3x}{7} = \dfrac{11 - 3y}{5}$

Raise the following expressions to the powers indicated:

19. $(-7a^2b^3)^3$. **21.** $(x-2)^4$. **23.** $(abc-4)^3$.

20. $(24a^7b^8)^2$. **22.** $(x+3)^5$. **24.** $(1-a-a^2)^2$.

Simplify the following expressions:

25. $\sqrt[4]{16a^8b^{12}c^{16}}$. **27.** $\sqrt[3]{-1728a^6b^9c}$. **29.** $\sqrt[6]{64a^6x^{12}}$.

26. $\sqrt[5]{32x^{10}y^{15}z^5}$. **28.** $\sqrt[4]{81a^8b^{12}c^{16}d}$. **30.** $\sqrt[7]{128m^7n^{14}p}$.

Find the square root of the following expressions:

31. $x^6-4x^5y+8x^4y^2-10x^3y^3+8x^2y^4-4xy^5+y^6$.

32. $x^6+25x^2+10x^4-4x^5-20x^3+16-24x$.

33. Find the square root of 5 to five decimal places.

Find the square root of the following numbers:

34. 120,409. **36.** 1867.1041. **38.** 64.128064.

35. 4816.36. **37.** 1435.6521. **39.** 16,803.9369.

Simplify the following expressions:

40. $\sqrt{125}$. **42.** $\sqrt[4]{729}$. **44.** $3\sqrt[6]{a^{12}b^{19}}$.

41. $\sqrt[3]{162}$. **43.** $6\sqrt[5]{a^{18}c^8}$. **45.** $\sqrt[3]{-1458}$.

46. Multiply $2\sqrt{x}-7$ by $3\sqrt{x}$.

47. Divide $\sqrt[3]{2}-\sqrt[3]{6}+\sqrt[3]{10}-\sqrt[3]{12}$ by $\sqrt[3]{2}$.

Divide the following:

48. $\dfrac{3}{\sqrt{7}+\sqrt{5}}$. **50.** $\dfrac{6}{5-2\sqrt{6}}$. **52.** $\dfrac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}$.

49. $\dfrac{7}{2\sqrt{5}-\sqrt{6}}$. **51.** $\dfrac{4-\sqrt{2}}{1+\sqrt{2}}$. **53.** $\dfrac{7+2\sqrt{10}}{7-2\sqrt{10}}$.

54. Extract the square root of $4+2\cdot 2\sqrt{3}+3$.

55. Solve the equation $x^2+x-a^2=x+2ab+b^2-\frac{1}{4}$.

56. Solve the equation $x(x+1)-b^2=x+a(a-2b)$.

Exercise 149. Review of Chapters I–XVI

Given $A = x^4 + b^4 - a^2x^2 + 2b^2x$, $B = x^4 - b^4 - 7a^2x^2 - 9b^2x$, $C = x^3 + b^3 + ax$, *perform the operations indicated:*

1. $A + B$. **3.** $7A - B$. **5.** AB. **7.** BC.

2. $A - B$. **4.** $7B - A$. **6.** AC. **8.** $A \div C$.

Factor the following expressions:

9. $(a^2 + 2b^2)^2 - a^2b^2$. **11.** $15x^2 - 7x - 2$.

10. $(2x - 3y)^2 - (x - 2y)^2$. **12.** $11x^2 - 54x + 63$.

Simplify the following expressions:

13. $\dfrac{3x - 2y}{3} - \dfrac{4y + 2x}{5} + \dfrac{22y - 9x}{15}$.

14. $\dfrac{3}{x - 2} + \dfrac{4a}{(x - 2)^2} - \dfrac{5a^2}{(x - a)^3}$.

15. A can do a piece of work in 10 da., and A and B together can do it in 7 da. In how many days can B do it alone?

Find the square root of:

16. 965.9664. **17.** $7 + 2\sqrt{21} + 3$.

18. $1 + 4x + 10x^2 + 12x^3 + 9x^4$.

Solve the following equations:

19. $x^2 + 4x = 12$. **22.** $x^2 - 7x = 8$. **25.** $x^2 - x = 6$.

20. $x^2 - 6x = 16$. **23.** $3x^2 - 4x = 7$. **26.** $5x^2 - 3x = 2$.

21. $x^2 - 12x = -5\frac{3}{4}$. **24.** $12x^2 + x = 1$. **27.** $2x^2 - 27x = 14$.

28. $5x(x - 3) - 2(x^2 - 6) = (x + 3)(x + 4)$.

29. Find the radius of a circle whose area would be doubled by increasing its radius 1 in. (Take $3\frac{1}{7}$ for π.)

30. Divide a line 20 in. long into two parts, such that the rectangle contained by the whole line and one part may be equal to the square on the other part.

Exercise 150. Review of Chapters I–XVII

Solve the following equations:

1. $x^2 - \frac{2}{3}x + \frac{1}{12} = 0$.
2. $\frac{3x}{4} + \frac{4}{3x} = \frac{13}{6}$.
3. $\frac{1}{2}x^2 - \frac{1}{3}x = 2(x+2)$.
4. $\frac{x+1}{x+4} = \frac{2x-1}{x+6}$.
5. $(x-2)(x-4) - 2(x-1)(x-3) = 0$.
6. $\frac{2x+3}{2(2x-1)} - \frac{7-x}{2(x+1)} = \frac{7-3x}{4-3x}$.
7. $x^2 - 3x - 3 - 6\sqrt{x^2 - 3x - 3} + 5 = 0$.

8. An iron bar weighs 36 lb. If it had been 1 ft. longer, with the same amount of iron, each foot would have weighed $\frac{1}{2}$ lb. less. Find the length of the bar and the weight per foot.

Solve the following equations, plotting each pair:

9. $x + y = 13$
   $xy = 36$
10. $x + y = 29$
    $xy = 100$
11. $x - y = 19$
    $xy = 66$
12. $x - y = 45$
    $xy = 250$
13. $x - y = 10$
    $x^2 + y^2 = 178$
14. $x - y = 9$
    $xy + 8 = 0$
15. $x - y = 1$
    $x^2 + y^2 = 8\frac{1}{2}$
16. $5x - 7y = 0$
    $20x^2 - 13xy = 16 - 28y^2$
17. $x - y = 7$
    $x^2 + xy + y^2 = 13$
18. $x^2 + xy = 35$
    $xy - y^2 = 6$
19. $xy - 12 = 0$
    $x - 2y = 5$
20. $xy - 7 = 0$
    $x^2 + y^2 = 50$
21. $2x - 5y = 9$
    $x^2 - xy + y^2 = 7$
22. $x^2 + 4xy = 3$
    $4xy + y^2 = 2\frac{1}{4}$
23. $x^2 - xy + y^2 = 48$
    $x - y - 8 = 0$
24. $x - 3y = 1$
    $xy + y^2 = 5$

Exercise 151. Review of Chapters I–XVIII

1. Define polynomial, and illustrate by three types.

2. The circumference of a circle is a function of what other line? Express your answer by a formula.

3. One factor of $x^4 + 4x^3 - 5x^2 - 36x - 36$ is $x - 3$, and another factor is 5 more than this. Find a third factor.

4. Find the prime factors of $x^6 - 2x^3 + 1$.

5. Find the number whose double increased by 24 exceeds 80 by as much as the number itself is less than 100.

6. What is the square of $x^2 - 5x + 7$?

7. Factor $121x^4 - 286x^2y^2 + 169y^4$.

8. One factor of $25a^4 - 9a^2b^2 + 16b^4$ is $5a^2 + 7ab + 4b^2$. Find the other factor.

Factor the following expressions:

9. $y^2 + 19yz + 48z^2$.

10. $a^4b^6 - 11a^2b^3 + 30$.

11. $c^{10} - 9c^5 - 10$.

12. $6b^2 - 7bx - 3x^2$.

13. $4x^2 + 8x + 3$.

14. $x^4 + x^3 + x^2 + x$.

Find the common factors of the following expressions:

15. $x^2 - y^2$, $x^3 - y^3$, and $x^2 - 7xy + 6y^2$.

16. $x^3 - 1$, $x^3 - 1$, $x^2 + x - 2$, and $2x^2 - 21x + 19$.

Find the factors of the following expressions:

17. $6(x^2 + xy)$, $8(xy - y^2)$, and $10(x^2 - y^2)$.

18. $x^2 + 11x + 30$, $x^2 + 13x + 42$, and $x^2 + 12x + 35$.

19. Multiply $\dfrac{x^2 - 2xy + y^2 - z^2}{x^2 + 2xy + y^2 - z^2}$ by $\dfrac{x + y - z}{x - y + z}$.

20. Simplify $\dfrac{2}{(x^2 - 1)^2} - \dfrac{1}{2x^2 - 4x + 2} - \dfrac{1}{1 - x^2}$.

21. Solve the equation $\dfrac{6x + 7}{15} - \dfrac{2x - 2}{7x - 6} = \dfrac{2x + 1}{5}$.

Solve the following equations:

22. $8x + 4y - 3z = 6$
$x + 3y - z = 7$
$4x - 5y + 4z = 8$

23. $12x + 5y - 4z = 29$
$13x - 2y + 5z = 58$
$17x - y - z = 15$

24. The sum of the two digits of a number is 10, and if 54 is added to the number, the digits will be interchanged. Find the number.

25. Multiply $\sqrt{7} + 3\sqrt{3}$ by $\sqrt{7} - 2\sqrt{3}$.

26. Divide $\sqrt{5} - \sqrt{6}$ by $2\sqrt{5} - \sqrt{6}$.

27. Extract the square root of $7 - 2\sqrt{14} + 2$.

Solve the following equations:

28. $\frac{x^2 - 5}{3} + \frac{2x^2 + 1}{6} = \frac{1}{2}$.

29. $\frac{3}{4x^2} - \frac{1}{6x^2} = \frac{7}{3}$.

30. $\sqrt{x+1} + \sqrt{x+16} = \sqrt{x+25}$.

31. $x^2 + xy + 2y^2 = 74$
$2x^2 + 2xy + y^2 = 73$

32. $x^2 + xy + 4y^2 = 6$
$3x^2 + 8y^2 = 14$

33. The hypotenuse of a right triangle is 20 and the area of the triangle is 96. Find the sides.

If $a : b = c : d$, prove that:

34. $ma : nb = mc : nd$.

35. $(a + 2b) : b = (c + 2d) : d$.

36. Find x when $x + 5 : 2x - 3 = 5x + 1 : 3x - 3$.

37. Find x when $\sqrt{x} + \sqrt{b} : \sqrt{x} - \sqrt{b} = a : b$.

38. Find x when $x : 48 = 75 : x$.

39. Find x when $216 : x = x : 864$.

40. If $a : b = c : d$, and $a = \sqrt{d}$, what is the ratio of a^2 to c?

41. If $a : b = b : c$, and $b^2 = c$, what is the value of a? If it is also given that $c = 9$, what are the values of a and b?

42. If $x : b = x + b : x - b$ show that x has two values, one positive and one negative.

43. One factor of $64x^4 + 128x^2y^2 + 81y^4$ is $8x^2 + 4xy + 9y^2$. What is the other factor?

44. Factor $a^2 - 2ax + x^2 + a - x$.

45. Factor $a^2 + 2ad + d^2 - 4b^2 + 12bc - 9c^2$.

46. Simplify $\dfrac{a-b}{(b+c)(c+a)} + \dfrac{b-c}{(c+a)(a+b)} + \dfrac{c-a}{(a+b)(b+c)}$.

47. Solve the equation $\dfrac{6x+13}{15} - \dfrac{3x+5}{5x-25} = \dfrac{2x}{5}$.

Solve the following equations:

48.
$$5x + 3y - 6z = 4$$
$$3x - y + 2z = 8$$
$$x - 2y + 2z = 2$$

50.
$$4x - 5y + 2z = 6$$
$$2x + 3y - z = 20$$
$$7x - 4y + 3z = 35$$

49.
$$x + \frac{y}{2} + \frac{z}{3} = 6$$
$$\frac{x}{3} + y + \frac{z}{2} = -1$$
$$\frac{x}{2} + \frac{y}{3} + z = 17$$

51.
$$\frac{1}{x} + \frac{2}{y} - 5 = 0$$
$$\frac{3}{y} - \frac{4}{z} + 6 = 0$$
$$\frac{4}{x} - \frac{1}{z} - 1 = 0$$

52. A sum of money at simple interest amounted to \$26,000 in 6 yr., and to \$30,000 in 10 yr. Find the sum and the rate.

53. What is the value of $(2^2)^2$? of 2^{2^2}? of $(2^2)^3$? of 2^{2^3}?

Find the square root of:

54. 1,283,689. **55.** 4,937,284. **56.** 1,522,756.

If $a : b = c : d$, *prove that:*

57. $a - b : a + b = c - d : c + d$.

58. $ma + b : ma - b = mc + d : mc - d$.

59. $a - nb : a + nb = c - nd : c + nd$.

60. $ma + nb : ma - nb = mc + nd : mc - nd$.

61. $2a + 3b : 3a - 4b = 2c + 3d : 3c - 4d$.

234. Highest Common Factor and Lowest Common Multiple. In the work of elementary algebra these two subjects play so small a part that many teachers now favor their omission. Because they are required in certain courses of study, however, they are given in this Appendix and may be studied in connection with factoring if desired.

235. Highest Common Factor. A factor of two or more expressions is called a *common factor* of the expressions.

For example, 2 is a common factor of 12 and 16, and $a-b$ is a common factor of $b-a$ and a^2-b^2, for $(b-a)\div(a-b)=-1$ and $(a^2-b^2)\div(a-b)=a+b$.

The factor of highest degree that is common to two or more expressions is called their *highest common factor* (*H.C.F.*).

For example, as 4 is the greatest common divisor of 12 and 16, so a^2-b^2 is the highest common factor of a^2-b^2 and a^4-b^4.

The reason that the subject is of so little importance in elementary algebra is that in reducing fractions to lowest terms we usually cancel the factors one at a time.

236. Finding the Highest Common Factor. In elementary algebra the highest common factor may be found by factoring.

1. Find the H.C.F. of $6x^2+7x-3$ and $4x^2-4x-15$.

$$6x^2+7x-3=(2x+3)(3x-1).$$
$$4x^2-4x-15=(2x+3)(2x-5).$$

Since $2x+3$ is the only common factor, it is the H.C.F.

2. Find the H.C.F. of ax^4-ay^4 and $a^2x^4-2a^2x^2y^2+a^2y^4$.

$$\begin{aligned} ax^4-ay^4 &= a(a^4-y^4)\\ &= a(x^2+y^2)(x^2-y^2)\\ &= a(x^2+y^2)(x+y)(x-y).\end{aligned}$$
$$\begin{aligned} a^2x^4-2a^2x^2y^2+a^2y^4 &= a^2(x^4-2x^2y^2+y^4)\\ &= a^2(x^2-y^2)^2\\ &= a^2(x+y)(x-y)(x+y)(x-y).\end{aligned}$$

The factors common to both expressions are a, $x+y$, and $x-y$. Therefore $a(x+y)(x-y)$ is the H.C.F.

Exercise 152. Highest Common Factor

Examples 1 to 12, oral — Examples 13 to 28, written

1. Name any common factor of 32 and 48.
2. Name any common factor of $15x^2y^2$ and $25xyz$.
3. What is the H.C.F. of m^3n and m^2n^2? of ab^3 and a^3b?

Find the H.C.F. of the following:

4. a^4b^4, a^2b^2.
5. a^6b^6, a^2b^4.
6. $a^2b^2c^2$, $a^3b^3c^3$.
7. $x^4y^4z^4$, $x^6y^4z^2$.
8. $x^5y^4z^3$, $x^4y^5z^2$.
9. $x^6y^4z^2$, $x^4y^5z^2$.
10. $a^3(b+c)$, a^2.
11. $a^2(b-c)$, a^3b^3.
12. a^2-b^2, $b-a$.
13. Find a common factor of $2a^2-3ab+b^2$, $3a^2-2ab-b^2$.

Find the H.C.F. of the following:

14. $6a^2-ab-b^2$, $3a^2+4ab+b^2$.
15. $6a^2+11ab+4b^2$, $2a^2+3ab+b^2$.
16. $30a^2+4a-2$, $6a^2+44a+14$.
17. $6a^2+7a+2$, $40a^2-70a-45$.
18. a^4-a^3b, ab^4-b^5, $7b^2-7a^2$.
19. $3a^8+3a^7b$, ab^8+b^4, a^2+b^2+2ab.
20. $a^2b^2+ab^3$, $a^4b+a^3b^2$, $9a^4(ab+2b^2)+9a^3b^3$.
21. $a(a-2b)+b^2$, b^2-a^2, a^3-b^3, $6a^4-6b^4$.
22. $a^2+b(b+2a)$, a^3-b^3, $a^2b^2+a^3(2b+a)$.
23. $x^4-2x^2y^2+y^4$, x^2+y^2+2xy, $12(x+y)^3$.
24. $7x^3+42xy^2+35x^2y$, $2x^3y+5x^2y^2+2xy^3$.
25. $2x^4y^2+7x^3y^3-9x^2y^4$, $6x^3y^3+33x^2y^4+27xy^5$.
26. $7x^2+21x+14$, $x^3+6x^2+11x+6$.

Reduce the following to lowest terms by canceling the H.C.F. from both numerator and denominator:

27. $\dfrac{y^4-x^4}{x^4-2x^2y^2+y^4}$.

28. $\dfrac{y^6-x^6}{(y+x)(x^2+xy+y^2)}$.

237. Multiple. An algebraic expression which contains another algebraic expression as a factor is called a *multiple* of that expression.

For example, 6 is a multiple of 2 and 3, and $a^2 - b^2$ is a multiple of $a + b$ and $a - b$.

238. Common Multiple. An algebraic expression which is a multiple of two or more expressions is called a *common multiple* of those expressions.

For example, 12 is a common multiple of 2, 3, 4, and 6. Likewise $a^4 - b^4$ is a common multiple of $a + b$ and $a - b$, of $a^2 - b^2$ and $a^2 + b^2$.

239. Lowest Common Multiple. The multiple of lowest degree that *contains* two or more algebraic expressions as factors is called their *lowest common multiple.*

The abbreviation for lowest common multiple is L.C.M.

For example, 12 is a common multiple of 2 and 3, but 6 is their L.C.M. Similarly, $a^4 - b^4$ is a common multiple of $a + b$ and $a - b$, but $a^2 - b^2$ is their L.C.M.

The L.C.M. is not necessarily the least common multiple for all values of the letters. Thus, if $a = 4$ and $b = 2$, the L.C.M. of $a + b$ and $a - b$, which is $a^2 - b^2$, reduces to $16 - 4 = 12$. The least common multiple of 6 and 2 is, however, 6.

In speaking of the L.C.M., a quantity is considered as a multiple of itself. Thus $a^2 - b^2$ is the L.C.M. of $a^2 - b^2$ and $a - b$. It is also the L.C.M. of $b^2 - a^2$ and $a - b$, the factor -1 not being considered because it does not affect the *degree.*

240. Finding the Lowest Common Multiple. Since the L.C.M. contains each quantity, it must contain all the factors of each, and since it is to be of lowest possible degree, it must contain no unnecessary factors. Therefore

The L.C.M. of two or more expressions must contain all the different factors of the expressions, each factor being taken the greatest number of times that it occurs in any of the given expressions.

For example, the L.C.M. of ab^2 and a^2b is a^2b^2, a and b each being taken twice as a factor.

1. Find the L.C.M. of $15\,a^3b$, $20\,ab^5$, and $30\,a^4b^4$.

$$15\,a^3b = 3 \cdot 5 \cdot a^3b,$$
$$20\,ab^5 = 2 \cdot 2 \cdot 5 \cdot ab^5,$$
$$30\,a^4b^4 = 2 \cdot 3 \cdot 5 \cdot a^4b^4.$$

The L.C.M. must contain each quantity, and hence it must contain all the factors of each quantity.

Since it is to be of the lowest degree, it can contain no factor of higher degree than occurs in any one of the quantities.

Therefore the $\text{L.C.M.} = 2 \cdot 2 \cdot 3 \cdot 5 \cdot a^4b^5$
$= 60\,a^4b^5$.

2. Find the L.C.M. of $6\,a^2+11\,ab+3\,b^2$ and $4\,a^2-4\,ab-15\,b^2$.

$$6\,a^2 + 11\,ab + 3\,b^2 = (2\,a + 3\,b)(3\,a + b).$$
$$4\,a^2 - 4\,ab - 15\,b^2 = (2\,a + 3\,b)(2\,a - 5\,b).$$
$$\therefore \text{ the L.C.M.} = (2\,a + 3\,b)(3\,a + b)(2\,a - 5\,b).$$

Exercise 153. The Lowest Common Multiple

Examples 1 to 3, oral — Examples 4 to 15, written

1. Find the L.C.M. of x^2y and x^2y^2; of $4\,mn^2$ and $6\,m^2n$.

2. Find the L.C.M. of $x - y$ and $x + y$; of $x^2 - y^2$ and $x - y$.

3. Find the L.C.M. of $2\,abc$, $5\,a^2b^2c^2$, and $10\,a^3b^3c^3$.

Find the L.C.M. of the following:

4. $a^2 - b^2$, $b + a$.

5. $a^2 - 1$, $a^2 + 1$.

6. $29\,m^2n^4x^9$, $174\,m^9n^4x^2$.

7. $a^4 - 1$, $a^2 - 1$.

8. $x^3 + x$, $x^4 - 1$.

9. $x^2 + 2\,x$, $x^2 - 4$.

10. $a^2 + 7\,ab + 10\,b^2$, $a^2 + 8\,ab + 15\,b^2$.

11. $m^2 + 16\,mn + 63\,n^2$, $m^2 - 81\,n^2$.

12. $x^4 - y^4$, $(y - x)^2$, $(x + y)^2$.

13. $x^2 - 2\,x - 63$, $x^2 + 7\,x + 10$, $x^2 - 7\,x - 18$.

14. $y - x$, $x^2 - y^2$, $x^3 - y^3$, $y^4 - x^4$.

15. $a^3 + 3\,a^2b + 3\,ab^2 + b^3$, $a^3 + b^3$, $a^4 - b^4$.

241. Binomial Theorem. There is occasionally required in the work of the first year in algebra the proof of the Binomial Theorem for the expansion of any power of a binomial:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots.$$

The proof may be taken in connection with special products or with powers and roots, or may be postponed until Book II is studied.

242. Factorial. The product of the positive integers from one to any given number n, inclusive, is called *factorial n.*

That is, $1\cdot 2\cdot 3\cdot 4 =$ factorial 4, and $1\cdot 2\cdot 3\cdots n =$ factorial n.

There are two common symbols for factorial n, as follows: $\lfloor\underline{n}$ and $n!$.

243. Binomial Theorem restated. In order to develop the Binomial Theorem we may write the expansion of the first three powers of $a+b$ in the following form:

$$(a+b)^1 = a+b;$$

$$(a+b)^2 = a^2 + 2ab + \frac{2\cdot 1}{2!}b^2;$$

$$(a+b)^3 = a^3 + 3a^2b + \frac{3\cdot 2}{2!}ab^2 + \frac{3\cdot 2\cdot 1}{3!}b^3.$$

We therefore infer that, in the case of $(a+b)^n$,

1. *The number of terms is greater by one than the exponent of the power to which the binomial is raised.*

2. *The exponent of a in the first term is n, and it decreases by one to the right.*

3. *The exponent of b in the first term is 0, and it increases by one to the right.*

4. *The coefficient of the first term is 1, and of the second term n.*

5. *The coefficient of each term after the first is found from the next preceding term by multiplying the coefficient of that term by the exponent of a and dividing the product by a number greater by one than the exponent of b.*

244. Proof of the Binomial Theorem. We know that the laws just stated hold for the third power, because we can obtain this power (§ 243) by actual multiplication.

Let us, for the moment, assume that they hold for the kth power, k being any positive integer. Then

$$(a+b)^k = a^k + ka^{k-1}b + \frac{k(k-1)}{2!}a^{k-2}b^2 + \frac{k(k-1)(k-2)}{3!}a^{k-3}b^3 + \cdots. \quad (1)$$

If we multiply both members of (1) by $a+b$ in the usual manner, we have

$$(a+b)^{k+1} = a^{k+1} + (k+1)a^k b + \frac{(k+1)k}{2!}a^{k-1}b^2 + \frac{(k+1)k(k-1)}{3!}a^{k-2}b^3 + \cdots. \quad (2)$$

But (2) is exactly what we obtain if we expand $(a+b)^{k+1}$ by the Binomial Theorem.

Therefore, *if* the law holds for the kth power (that is, if (1) is true), it holds for the $(k+1)$th power, for we have shown this by actually multiplying (1) by $a+b$.

But the law *does* hold true for the *third* power (§ 243), and therefore it must hold true for the $(3+1)$th, or *fourth* power. Since it holds true for the fourth power, it must hold true for the $(4+1)$th, or fifth power, and so on for any positive integral power. Therefore, for the nth power,

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots.$$

This is a method of proof known as *mathematical induction.*

Evidently we may interchange a and b. This will give us the last few terms of the series, just as we have now the first few, thus:

$$\cdots + \frac{n(n-1)}{2!}a^2b^{n-2} + nab^{n-1} + b^n.$$

This is illustrated in such a familiar case as $a^3 + 3a^2b + 3ab^2 + b^3$.

1. Expand $(a-b)^n$.

Evidently the even powers of $-b$ are positive (§ 168) and the odd powers negative. We therefore have

$$(a-b)^n = a^n - na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 - \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots.$$

We cannot tell the sign of the last term unless we know whether n is odd or even. It is, however, $(-b)^n$.

2. Expand $(1+x)^n$.

Substituting 1 for a, and x for b, in $(a+b)^n$, we have

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + \frac{n(n-1)}{2!}x^{n-2} + nx^{n-1} + x^n.$$

3. Expand $(1-x)^n$.

In Ex. 2 put $-x$ for x, and we have

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \cdots.$$

We cannot tell the sign of the last term unless we know whether n is odd or even. It is, however, $(-x)^n$.

4. Expand $(1+2a)^6$.

Substituting 1 for a, and $2a$ for b, in $(a+b)^n$, we have

$$(1+2a)^6 = 1 + 6(2a) + \frac{6\cdot5}{2!}(2a)^2 + \frac{6\cdot5\cdot4}{3!}(2a)^3 + \frac{6\cdot5\cdot4\cdot3}{4!}(2a)^4 + \frac{6\cdot5\cdot4\cdot3\cdot2}{5!}(2a)^5 + \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{6!}(2a)^6$$
$$= 1 + 12a + 60a^2 + 160a^3 + 240a^4 + 192a^5 + 64a^6.$$

5. Expand to four terms $\left(\frac{2}{x} - \frac{3x^2}{4}\right)^7$.

Substituting $\frac{2}{x}$ for a, and $-\frac{3x^2}{4}$ for b, in $(a+b)^n$, we have

$$\left(\frac{2}{x} - \frac{3x^2}{4}\right)^7 = \left(\frac{2}{x}\right)^7 - 7\left(\frac{2}{x}\right)^6\left(\frac{3x^2}{4}\right) + 21\left(\frac{2}{x}\right)^5\left(\frac{3x^2}{4}\right)^2 - 35\left(\frac{2}{x}\right)^4\left(\frac{3x^2}{4}\right)^3 + \cdots$$
$$= \frac{128}{x^7} - \frac{336}{x^4} + \frac{378}{x} - \frac{945x^2}{4} + \cdots.$$

We could also have written this $(2x^{-1} - \frac{3}{4}x^2)^7$ and expanded, obtaining $128x^{-7} - 336x^{-4} + 378x^{-1} - \frac{945}{4}x^2 + \cdots$.

245. Formula for the rth term. We see by § 243 that the third term is $\dfrac{n(n-1)}{2!}a^{n-2}b^2$,

the fourth term is $\dfrac{n(n-1)(n-2)}{3!}a^{n-3}b^3, \ldots$

and hence that the rth term is

$$\frac{n(n-1)(n-2)\cdots \text{to } (r-1) \text{ factors}}{(r-1)!}a^{n-(r-1)}b^{r-1}.$$

Find the eighth term of $\left(4-\dfrac{x^2}{2}\right)^{10}$.

Here $a=4,\ b=-\frac{1}{2}x^2,\ n=10,\ r=8.$

The eighth term is

$$\frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4}{7!}(4)^{10-7}(-\tfrac{1}{2}x^2)^7, \text{ or } -60x^{14}.$$

Exercise 154. Binomial Theorem

Expand the following expressions:

1. $(1+x)^5$.
2. $(1-x)^7$.
3. $(x+2)^7$.
4. $(x-2)^8$.
5. $(a+2b)^4$.
6. $(a-2b)^5$.
7. $(\frac{1}{2}a+\frac{2}{3}b)^4$.
8. $\left(\dfrac{2}{x}-\dfrac{x}{2}\right)^5$.
9. $(x+3)^5$.
10. $(2x-1)^4$.
11. $(1-3x)^5$.
12. $(4x-3)^4$.
13. $(a+2b)^6$.
14. $(2a-3b)^6$.
15. $(\frac{1}{2}a+\frac{2}{3}b)^6$.
16. $\left(\dfrac{3}{x}-\dfrac{x^2}{5}\right)^6$.
17. $(2x+1)^6$.
18. $(1-2x)^5$.
19. $(\frac{1}{2}+x)^8$.
20. $(x-\frac{1}{3})^8$.
21. $(a^{\frac{1}{2}}+b^{\frac{1}{2}})^8$.
22. $(2a^{\frac{1}{2}}-x^{\frac{1}{2}})^6$.
23. $(\frac{1}{2}\sqrt{x}-\frac{2}{3}\sqrt{y})^6$.
24. $\left(\sqrt[3]{\dfrac{x^2}{3}}-\sqrt{\dfrac{2}{x}}\right)^8$.
25. In $(3x-4y)^8$ find the fifth term; the sixth.
26. In $(2x+3)^{12}$ find the ninth term; the tenth.
27. In $(3x-\frac{1}{2}y^2)^{18}$ find the tenth term; the twelfth.
28. In $(4a-3b^2)^{14}$ find the middle term.
29. In $(a^{\frac{1}{2}}+b^{\frac{1}{3}})^{10}$ find the middle term.

246. Variation. If two variable quantities, x and y, have a constant ratio k, either quantity is said to *vary* as the other.

If $\frac{x}{y} = k$, then $x = ky$, and $y = \frac{1}{k} \cdot x$.

The expression "x varies as y" is sometimes written $x \propto y$, which means that $x = ky$, k being the constant ratio.

For example, in a circle $c = 2\pi r$, so that $c \propto r$, c always being 2π times r.

Similarly, for the area of a circle, $a = \pi r^2$, in which case $a \propto r^2$.

If $x \propto y$, and $x = 12$ when $y = 4$, find x when $y = 6$.

Since $x \propto y$, therefore $x = ky$, where k is some constant.

Hence $12 = k \cdot 4$,

and therefore $3 = k$.

Hence, when $y = 6$,

$x = k \cdot 6 = 3 \cdot 6 = 18$.

Exercise 155. Variation

1. If $x = ky$, find k when $x = 23$ and $y = 11.5$.
2. If $x = ky$, find y when $x = 13.5$ and $k = 2.25$.
3. If $a = \pi r^2$, find a when $\pi = 3.1416$ and $r = 6$.
4. If $x \propto y$, and $x = 38$ when $y = 14$, find x when $y = \frac{3}{4}$.
5. If $x \propto y$, and $x = 13$ when $y = 7$, find y when $x = \frac{3}{5}$.
6. If $x \propto y$, and the ratio of x to y is $\frac{2}{3}$, find x when $y = 27$.
7. In Ex. 6 find y when $x = 16$; when $x = 124$.
8. If $x = ky$ and $y = mz$, find x in terms of z.
9. If $x \propto y$ and $y \propto z$, show that $x \propto z$.
10. If $x \propto y$ and $y = z$, what follows?
11. If $a \propto r^2$, and $a = 3.1416$ when $r = 1$, what is the value of a when $r = 9$?
12. If the circumference of a circle varies as the radius, and the radius varies as the diameter, show that the circumference varies as the diameter.

247. Joint Variation. The area of a triangle varies as the base when the altitude is constant, and as the altitude when the base is constant. If both base and altitude vary, the area varies as their product. This illustrates *joint variation.*

248. Inverse Variation. If two variables, x and y, are so related that the ratio of x to $\frac{1}{y}$ is some constant number, k, either quantity is said to *vary inversely* as the other.

If $x : \frac{1}{y} = k$, then $xy = k$, or $x = \frac{k}{y}$ and $y = \frac{k}{x}$.

As y increases, x decreases at the same rate. If y is made twice as large, x becomes half as large.

For example, if a certain number of men, m, can do a piece of work in t days, then twice as many men can do the work in $\frac{1}{2}t$ days, and $\frac{1}{4}$ as many men can do it in $4t$ days. That is, $m \propto \frac{1}{t}$.

If $x \propto y$ and $x \propto \frac{1}{z}$, and $x = 12$ when $y = 6$ and $z = 2$, find x when $y = 8$ and $z = 1$.

Since $x \propto y$ and $x \propto \frac{1}{z}$, we have $x = ky \cdot \frac{1}{z}$.

Hence $12 = k \cdot 6 \cdot \frac{1}{2}$, and therefore $k = 4$.

Hence $x = 4 \cdot 8 \cdot 1 = 32$, when $y = 8$ and $z = 1$.

Exercise 156. Inverse and Joint Variation

1. If x varies inversely as y, and $x = 12$ when $y = 2$, find x when $y = 7$.

2. If $x \propto y$ and $x \propto \frac{1}{z}$, and if $x = 6$ when $y = 8$ and $z = 12$, find x when $z = 28$ and $y = 4$.

3. If $x \propto y$ and $x \propto \frac{1}{z}$, and if $x = 15$ when $y = 3$ and $z = 5$, find y when $x = 35$ and $z = 12.5$.

4. If $y \propto \frac{1}{x}$ and $y \propto \frac{1}{z}$, and if $x = 2$ when $y = 4$ and $z = 3$, find z when $x = 4$ and $y = 1$.

249. The Laws of Fractions. If desired, the usual laws of fractions may be presented with less regard to arithmetic than was done in Chapter XI, as follows:

1. *Multiplying both numerator and denominator of a fraction by the same expression does not change the value of the fraction.*

If	$\frac{a}{b} = x,$	
then	$a = bx,$	Axiom 3
and	$na = nbx.$	Axiom 3
Therefore	$\frac{na}{nb} = x.$	Axiom 4
But	$\frac{a}{b} = x,$	
and hence	$\frac{a}{b} = \frac{na}{nb}.$	Axiom 6

2. *Dividing both numerator and denominator of a fraction by the same expression does not change the value of the fraction.*

For it has just been proved that

$$\frac{na}{nb} = \frac{a}{b}.$$

3. *Multiplying the numerator by any number multiplies the fraction by that number, and dividing the numerator by any number divides the fraction by that number.*

Let x be the value of the fraction $\frac{a}{b}$.

Then	$\frac{a}{b} = x,$	
and	$a = bx,$	
and	$na = nbx.$	Axiom 3
Then	$\frac{na}{b} = nx.$	Axiom 4

That is, multiplying a by n multiplies x, the value of the fraction, by n.

Likewise, since	$a = bx,$
therefore	$a \div n = bx \div n,$
and	$\frac{a \div n}{b} = \frac{bx \div n}{b} = x \div n.$

That is, dividing a by n divides x, the value of the fraction, by n.

4. *Multiplying the denominator by any number divides the fraction by that number, and dividing the denominator by any number multiplies the fraction by that number.*

Let $\frac{a}{b} = x.$

Then $a = bx,$

and $a = \frac{nbx}{n}.$ Law 1

Therefore $\frac{a}{nb} = \frac{x}{n}.$ Axiom 4

That is, multiplying b by n divides x, the value of the fraction, by n. The proof for division by n is substantially the same.

5. *To multiply one fraction by another, take the product of the numerators for a numerator, and the product of the denominators for a denominator.*

Let the two fractions be $\frac{a}{b}$ and $\frac{m}{n}$.

Let $\frac{a}{b} = x,$ whence $a = bx;$

and let $\frac{m}{n} = y,$ whence $m = ny.$

Then, multiplying, $am = bnxy.$ Axiom 3

Hence, dividing, $\frac{am}{bn} = xy.$ Axiom 4

That is, xy, which is the product of the fractions, is found by taking am for the numerator and bn for the denominator.

6. *To divide one fraction by another, multiply the dividend by the reciprocal of the divisor.*

Let $\frac{a}{b} \div \frac{m}{n} = x.$

Then $\frac{a}{b} = x \cdot \frac{m}{n} = \frac{xm}{n}.$ Axiom 3

Then $\frac{an}{b} = xm,$ Axiom 3

and $\frac{an}{bm} = x.$ Axiom 4

But $\frac{an}{bm}$ is the same as $\frac{a}{b}$ multiplied by the reciprocal of $\frac{m}{n}$.

250. Factoring the General Quadratic Trinomial. As stated on page 108, there are several methods of factoring a trinomial of the type $ax^2 + bx + c$. The teacher may substitute one of the following for the trial method already given:

1. *By splitting the middle term.* Since

$$(mx + n)(px + q) = mpx^2 + (np + mq)x + qn,$$

we see that the coefficient of x is the sum of two numbers (np and mq) whose product is the product of the coefficient of x^2 (that is, mp) and the absolute term (that is, qn).

Consider the trinomial $10x^2 - x - 21$.

What are the two numbers whose sum is -1 and whose product is $10(-21)$, or -210? Evidently -15 and 14.

Then
$$\begin{aligned} 10x^2 - x - 21 &= 10x^2 + 14x - 15x - 21 \\ &= 2x(5x + 7) - 3(5x + 7) \\ &= (2x - 3)(5x + 7). \end{aligned}$$

That is, to factor the type $ax^2 + bx + c$,

Find two numbers whose algebraic sum is b and product ac. Separate b into these parts and factor by grouping.

2. *By making the first term a square.* We proceed as follows:

$$\begin{aligned} 10x^2 - x - 21 &= \frac{100x^2 - 10x - 210}{10} \\ &= \frac{(10x)^2 - (10x) - 210}{10} \\ &= \frac{(10x - 15)(10x + 14)}{10} \\ &= \frac{5(2x - 3) \cdot 2(5x + 7)}{10} \\ &= (2x - 3)(5x + 7). \end{aligned}$$

3. *By substitution.* Taking the above example:

Let $y = 10x$, or $\frac{1}{10}y = x$.

Then
$$\begin{aligned} 10x^2 - x - 21 &= \tfrac{1}{10}y^2 - \tfrac{1}{10}y - 21 \\ &= \tfrac{1}{10}(y - 15)(y + 14) \\ &= \tfrac{1}{10}(10x - 15)(10x + 14) \\ &= (2x - 3)(5x + 7). \end{aligned}$$

4. *By formula.* It is possible to factor an expression of the type $ax^2 + bx + c$ by formula, without any trial.

$$ax^2 + bx + c = \frac{a^2x^2 + bax + ac}{a}$$

$$= \frac{(ax)^2 + b(ax) + \frac{b^2}{4} + ac - \frac{b^2}{4}}{a}$$

$$= \frac{\left(ax + \frac{b}{2}\right)^2 - \left(\frac{b^2}{4} - ac\right)}{a}$$

$$= \frac{\left(ax + \frac{b}{2} + \sqrt{\frac{b^2}{4} - ac}\right)\left(ax + \frac{b}{2} - \sqrt{\frac{b^2}{4} - ac}\right)}{a}.$$

For example, consider $10x^2 - x - 21$.

Here $a = 10$, $b = -1$, $c = -21$. Hence the factors are

$$\frac{(10x - \frac{1}{2} + \sqrt{\frac{1}{4} + 210})(10x - \frac{1}{2} - \sqrt{\frac{1}{4} + 210})}{10}$$

$$= \frac{(10x - \frac{1}{2} + \sqrt{8\frac{4}{4}\frac{1}{0}})(10x - \frac{1}{2} - \sqrt{8\frac{4}{4}\frac{1}{0}})}{10}$$

$$= \frac{(10x - \frac{1}{2} + \frac{2}{2}\frac{9}{0})(10x - \frac{1}{2} - \frac{2}{2}\frac{9}{0})}{10}$$

$$= \frac{(10x + 14)(10x - 15)}{10} = (5x + 7)(2x - 3).$$

5. *By quadratics.* After studying quadratic equations the class may proceed as follows in the factoring of $10x^2 - x - 21$:

Consider what values of x will make $10x^2 - x - 21 = 0$.

If $10x^2 - x - 21 = 0$,

then $x^2 - \frac{1}{10}x + \frac{1}{400} = \frac{21}{10} + \frac{1}{400} = \frac{841}{400}$.

$\therefore x - \frac{1}{20} = \pm \frac{29}{20}$,

and $x = \frac{1}{20} \pm \frac{29}{20} = \frac{3}{2}$ or $-\frac{7}{5}$

Hence $x - \frac{3}{2} = 0$, or $x + \frac{7}{5} = 0$,

and $(x - \frac{3}{2})(x + \frac{7}{5}) = 0$,

or $(2x - 3)(5x + 7) = 0 = 10x^2 - x - 21$.

An advanced class may study all five of these methods, using the examples on page 109.

251. Equations solved by Factoring. If we have the equation $(x-2)(2x+3)=0$, either one of the factors may equal zero and the equation is thereby satisfied.

For if $x-2=0$, then $0\cdot(2x+3)$ must equal zero;
and if $2x+3=0$, then $(x-2)\cdot 0$ must also equal zero.
But if $x-2=0$, then $x=2$;
and if $2x+3=0$, then $2x=-3$, and $x=-\frac{3}{2}$.

When the second member of an equation is zero, and the first member can be expressed as a product of two or more binomial factors, the equation can be solved by equating each of these factors to zero and solving the resulting simple equations.

This work may be introduced, if time allows, after page 114. It is reviewed in Book II, and may, if desired, be taken in that book.

1. Solve the equation $x^2-2x-35=0$.

If $x^2-2x-35=0$,
then $(x-7)(x+5)=0$.
Therefore $x-7=0$, and $x=7$;
or $x+5=0$, and $x=-5$.

These results can be checked, for $7^2-2\cdot 7-35=0$.
Similarly, $(-5)^2-2\cdot(-5)-35=0$.

2. Solve the equation $x-3\sqrt{x}=0$.

Since $x-3\sqrt{x}=0$,
we see that $\sqrt{x}(\sqrt{x}-3)=0$.
Therefore $\sqrt{x}=0$, and $x=0$;
or $\sqrt{x}-3=0$, $\sqrt{x}=3$, and $x=9$.

3. Solve the equation $x^3-36x=0$.

Since $x^3-36x=0$,
we see that $x(x^2-36)=0$.
Therefore $x=0$;
or $x^2-36=0$,
whence $(x-6)(x+6)=0$,
and $x-6=0$, and $x=6$;
or $x+6=0$, and $x=-6$.

Exercise 157. Equations solved by Factoring

Solve the following equations by factoring:

1. $x^2 - 9x + 14 = 0$.
2. $x^2 - 10x + 21 = 0$.
3. $x^2 - 12x + 35 = 0$.
4. $x^2 - 13x + 40 = 0$.
5. $x^2 - 14x + 45 = 0$.
6. $x^2 - 4x - 45 = 0$.
7. $x^2 - 6x - 55 = 0$.
8. $x^2 - 7x - 78 = 0$.
9. $x^2 - 6x - 135 = 0$.
10. $2x^2 - 11x - 21 = 0$.
11. $6x^2 - 5x - 21 = 0$.
12. $8x^2 + 14x = 15$.
13. $15x^2 + 4x = 35$.
14. $56x^2 + x = 1$.
15. $2x^2 - 5x = 63$.
16. $4x^2 + 4x = 63$.
17. $12x^2 - 25x = 7$.
18. $12x^2 - 13x = 35$.
19. $132x^2 - 21x = 3$.
20. $143x^2 - 2x = 1$.
21. $x^3 - 4x = 0$.
22. $x^3 - 25x = 0$.
23. $x^2 - 25x = -156$.
24. $x^2 - 25x = 26$.
25. $(x^2 - 1)(x^2 - 4) = 0$.
26. $x^4 - 5x^2 + 4 = 0$.
27. $x^4 - 10x^2 + 9 = 0$.
28. $x^4 - 13x^2 + 36 = 0$.
29. $x^5 - 13x^3 + 36x = 0$.
30. $x^4 - 8x^3 + 7x^2 = 0$.
31. $6x^3 + x^2 - x = 0$.
32. $6x^4 - 7x^3 + 2x^2 = 0$.
33. $15x^3 + 32x^2 = 7x$.
34. $21x^3 - 8x = 22x^2$.
35. $132x^3 + x = 23x^2$.
36. $132x^4 + 3x^2 = 47x^3$.
37. $21x^3 + 64x^2 = -3x$.
38. $19x^3 + 56x^2 = 3x$.
39. $7x^3 + 7x = 14x^2$.
40. $9x^3 + 36x = 36x^2$.

41. If one root of the equation $x^3 - 6x^2 + 11x - 6 = 0$ is 3, what are the other two roots?

42. If one root of the equation $x^3 - 7x^2 + 14x - 8 = 0$ is 2, what are the other two roots?

43. If one root of the equation $x^4 - 12x^3 + 49x^2 - 78x + 40 = 0$ is 1, and another root is 2, what are the other roots?

44. Two roots of the equation $x^4 - 13x^3 + 53x^2 - 83x + 42 = 0$ are 1 and 7. Find the other roots.

252. History of Algebra. Problems such as we solve by algebra are very old. We find them in a book written in Egypt over 4000 years ago, and afterwards copied by one Ahmes about 1700 B.C. The Ahmes copy, made on a kind of paper called papyrus, is still preserved in the British Museum. It contains problems such as, "Mass, its seventh, its whole, it makes nineteen." We should write this: $\frac{1}{7}x + x = 19$. But Ahmes knew nothing of our algebraic symbols. He had a symbol for "mass," and rude symbols for addition, subtraction, and equality; but an algebraic equation, such as we use, was unknown to the ancient world.

We also find, in a later Egyptian manuscript, a problem that we would solve by quadratics. It is of the form $x + y = a$ and $xy = b$. Nothing but the answer is given, so we do not know how it was solved, but we know that algebra in our sense of the word was not used.

Euclid, who wrote the first great textbook on geometry, and who lived at Alexandria, in Egypt, about 300 B.C., knew and proved $(a + b)^2 = a^2 + 2ab + b^2$, $(a + b)(a - b) = a^2 - b^2$, and similar relations, but all this was proved by geometric figures similar to that on page 89, § 87. This was a cumbersome method, but the ancients were skillful in geometry and so were able to attain results that we now seek by algebra.

Archimedes of Syracuse, in Sicily, about 250 B.C., and others of his time, knew that $c = 2\pi r$ and that $a = \pi r^2$ (§§ 15, 16), but these statements had to be written out in words, the ancients having no good symbols like ours.

The first writer who seems to have developed an algebraic symbolism of any value was Diophantus, who lived in Alexandria about 275 A.D. He had symbols for the unknown quantity, its square, its cube, and so on to the sixth power. He also had a symbol for subtraction and one for equality, and his equations, while written in Greek, were somewhat like ours. He also knew that $-a$ times $-b$ equals $+ab$. He was much interested in indeterminate equations.

It is in the Orient that algebra as we know it had its beginning. The Hindus were much interested in fanciful problems, and some that we still find in our algebras seem to have been first suggested by them. The earliest of these writers was Aryabhatta (about 525 A.D.), who lived at Patna, on the Ganges River. He knew how to solve quadratic equations and had some knowledge of series.

The next great Hindu writer was Brahmagupta (about 650). He lived at Ujjain, formerly a great place for the study of astronomy, in west-central India. He knew how to solve quadratic equations and was also interested in the solution of indeterminate equations.

The third Hindu algebraist of note was Mahavir the Learned (Mahaviracarya), who lived in Mysore, in southern India, about 850. He wrote an extensive treatise on mathematics, which has recently been translated. He had no symbols for operations, and, as was the custom in India, wrote his entire work in verse. The nature of his problems, all of which were rather fanciful, may be inferred from the following: "A pile of apples divided among 2, 3, 4, or 5 persons leaves 1 as a remainder in each case. O you who know arithmetic, tell me the numerical measure of the pile."

The last great Hindu writer, before the European influence began to be manifest, was Bhaskara (about 1150). He had a daughter by the name of Lilavati, and he named his arithmetic after her. His algebra, the Bija Ganita as he called it, contained a number of symbols, but the equations were mostly written out in full. He knew the rules of signs, and that $a \div 0 = \infty$, and could, like his predecessors, solve a quadratic equation.

Meantime the Arabs, chiefly at Bagdad, beginning about 800, made much of algebra. Mohammed ibn Musa al Khowarizmi (Mohammed the son of Moses, the man from Khwarezm — the country about Khiva) wrote a work entitled *Al-jabr wa'l muqābalah* ("reuniting and comparison").

In this the unknown quantity is called the thing, or root. The book contains a very complete discussion of quadratics together with some treatment of surds. It became known in Europe in the Middle Ages, and hence the science went by such names as *Algebra*, *Almucabela*, and *Mucabel*, and the unknown quantity by such names as the Latin *res* ("thing") and the Italian *cosa* ("thing"). From the latter word the science was at one time called the Cossic Art in England and the Coss in Germany.

In eastern Persia the poet Omar Khayyam (about 1050) wrote an algebra in which there is some attempt to solve equations of the third degree of an easy nature, and one of the fourth degree, $(100 - x^2)(10 - x)^2 = 8100$.

One of the early European writers on algebra was Leonardo of Pisa (about 1200), or Leonardo Fibonacci. He wrote on arithmetic, algebra, and geometry, and was able to solve the equation $x^3 + 2x^2 + 10x - 20 = 0$, giving a result equal to 1.3688081075, a remarkable achievement for the time. While yet a boy he was taken by his father to the north coast of Africa, and there studied under a Moorish schoolmaster. At this time the Hindu-Arabic numerals, the ones that we ordinarily use, were known among the Moors, but were not yet common in Europe. Upon his return to Italy as a young man Leonardo wrote some books on mathematics and did much to make these numerals better known in Europe. He was one of the best mathematicians of his time.

Our present algebraic symbolism was mostly invented between about 1500 and 1650. The symbols + and − first appeared in print in Johann Widman's arithmetic (1489), the + apparently being suggested by the written form of the Latin *et* ("and"). The Italians at that time commonly used $\bar{p}$ and $\overline{m}$ for these purposes. The first noteworthy printed work to contain algebra was Paciolo's treatise of 1494. He uses *co* (Italian *cosa*, "thing") and ℞ (Latin *res*) for the unknown quantity, *ce* or Z (*census*) for x^2, *cu* or C (*cubus*) for x^3, *ce di ce* (*censo di censo*) for x^4, and so on, and p and $\overline{m}$ for plus and

minus. Tartaglia, a great Italian algebraist, who seems to have been the first to solve completely a general equation of the third degree, and who published a well-known treatise in 1556, used *22 men* (*22 men* R̃ *6* for $22-(22-\sqrt{6})$, and similar expressions. Cardan, his great rival, published Tartaglia's solution of the cubic equation in 1545. He wrote the equation $x^3+6x=20$ thus: cub^9 p : 6 reb^9 æq̄lis 20. Thus we see that the algebraists of that period had none of the convenient symbolism that we use to-day. For square root Paciolo used R̃ · 2^a, for cube root R̃ · 3^a, and so on. Stifel, a German mathematician of the sixteenth century, edited a work by Christoff Rudolf (1553), and in this he uses our present root signs. The use of x and y for unknown quantities, and a, b, $\cdots$, for known quantities, is due to the great French mathematician Descartes (1637). To him we also owe our graphs of equations, and he did much, through graphic representations, to make the negative number better understood. Our sign of equality was first used by an English mathematician, Robert Recorde, in 1557. Our symbols of aggregation started with the Italians in the sixteenth century. Bombelli (1572), for example, uses R.q ⌊128. *p.* 8^2⌋ for $\sqrt{128+\sqrt{8}}$, where ⌊ stands for *legato* ("bound"); and Cardan uses the expression R̃v: cu. R̃ 108 p: 10 for $\sqrt{\sqrt[3]{108}+10}$, the *v* standing for *universalis*. The symbols $>$ and $<$ were first used by an English algebraist, Harriot, in 1631. This name has particular interest to Americans because Harriot was sent to this country by Sir Walter Raleigh to make a survey of Virginia.

After about 1650 elementary algebra as we know it was substantially complete, except as new applications have come in, and for the past two hundred fifty years it has been the basis of all advanced mathematics. The chief addition to the elementary field has been the better understanding of the complex number, and this is due in large part to De Moivre (1730), Euler (1748), Wessel (1797), and Gauss (about 1830).

253. Table of Powers and Roots. The following table will be found helpful in plotting and solving equations:

No.	Squares	Cubes	Square Roots	Cube Roots	No.	Squares	Cubes	Square Roots	Cube Roots
1	1	1	1.000	1.000	51	2,601	132,651	7.141	3.708
2	4	8	1.414	1.259	52	2,704	140,608	7.211	3.732
3	9	27	1.732	1.442	53	2,809	148,877	7.280	3.756
4	16	64	2.000	1.587	54	2,916	157,464	7.348	3.779
5	25	125	2.236	1.709	55	3,025	166,375	7.416	3.802
6	36	216	2.449	1.817	56	3,136	175,616	7.483	3.825
7	49	343	2.645	1.912	57	3,249	185,193	7.549	3.848
8	64	512	2.828	2.000	58	3,364	195,112	7.615	3.870
9	81	729	3.000	2.080	59	3,481	205,379	7.681	3.892
10	100	1,000	3.162	2.154	60	3,600	216,000	7.745	3.914
11	121	1,331	3.316	2.223	61	3,721	226,981	7.810	3.936
12	144	1,728	3.464	2.289	62	3,844	238,328	7.874	3.957
13	169	2,197	3.605	2.351	63	3,969	250,047	7.937	3.979
14	196	2,744	3.741	2.410	64	4,096	262,144	8.000	4.000
15	225	3,375	3.872	2.466	65	4,225	274,625	8.062	4.020
16	256	4,096	4.000	2.519	66	4,356	287,496	8.124	4.041
17	289	4,913	4.123	2.571	67	4,489	300,763	8.185	4.061
18	324	5,832	4.242	2.620	68	4,624	314,432	8.246	4.081
19	361	6,859	4.358	2.668	69	4,761	328,509	8.306	4.101
20	400	8,000	4.472	2.714	70	4,900	343,000	8.366	4.121
21	441	9,261	4.582	2.758	71	5,041	357,911	8.426	4.140
22	484	10,648	4.690	2.802	72	5,184	373,248	8.485	4.160
23	529	12,167	4.795	2.843	73	5,329	389,017	8.544	4.179
24	576	13,824	4.898	2.884	74	5,476	405,224	8.602	4.198
25	625	15,625	5.000	2.924	75	5,625	421,875	8.660	4.217
26	676	17,576	5.099	2.962	76	5,776	438,976	8.717	4.235
27	729	19,683	5.196	3.000	77	5,929	456,533	8.774	4.254
28	784	21,952	5.291	3.036	78	6,084	474,552	8.831	4.272
29	841	24,389	5.385	3.072	79	6,241	493,039	8.888	4.290
30	900	27,000	5.477	3.107	80	6,400	512,000	8.944	4.308
31	961	29,791	5.567	3.141	81	6,561	531,441	9.000	4.326
32	1,024	32,768	5.656	3.174	82	6,724	551,368	9.055	4.344
33	1,089	35,937	5.744	3.207	83	6,889	571,787	9.110	4.362
34	1,156	39,304	5.830	3.239	84	7,056	592,704	9.165	4.379
35	1,225	42,875	5.916	3.271	85	7,225	614,125	9.219	4.396
36	1,296	46,656	6.000	3.301	86	7,396	636,056	9.273	4.414
37	1,369	50,653	6.082	3.332	87	7,569	658,503	9.327	4.431
38	1,444	54,872	6.164	3.361	88	7,744	681,472	9.380	4.447
39	1,521	59,319	6.244	3.391	89	7,921	704,969	9.433	4.464
40	1,600	64,000	6.324	3.419	90	8,100	729,000	9.486	4.481
41	1,681	68,921	6.403	3.448	91	8,281	753,571	9.539	4.497
42	1,764	74,088	6.480	3.476	92	8,464	778,688	9.591	4.514
43	1,849	79,507	6.557	3.503	93	8,649	804,357	9.643	4.530
44	1,936	85,184	6.633	3.530	94	8,836	830,584	9.695	4.546
45	2,025	91,125	6.708	3.556	95	9,025	857,375	9.746	4.562
46	2,116	97,336	6.782	3.583	96	9,216	884,736	9.797	4.578
47	2,209	103,823	6.855	3.608	97	9,409	912,673	9.848	4.594
48	2,304	110,592	6.928	3.634	98	9,604	941,192	9.899	4.610
49	2,401	117,649	7.000	3.659	99	9,801	970,299	9.949	4.626
50	2,500	125,000	7.071	3.684	100	10,000	1,000,000	10.000	4.641

INDEX

	PAGE
Abscissa	171
Absolute term	38, 211
value	25
Addition	28, 41, 121, 200
Affected quadratic	211
Aggregation, symbols of	6, 58
Algebraic expression	37
fraction	115
sum	42
Alternation, proportion by	241
Arranging a polynomial	65
Artificial numbers	27
Axes	170
Axioms	20
Binomial	6
Binomial Theorem	280
Check	44
Clearing of fractions	131
Coefficient	38
Coefficients of an equation	211
Common denominator	118
factor	276
Comparison	149
Complete quadratic	211
Completing the square	216
Complex number	210
Composition, proportion by	241
Conjugate surds	206
Constant	175, 211
Coördinate paper	171
Coördinates	171
Cube	5
Degree	60, 61, 173
Division	69, 129, 204
Division, proportion by	241
Elimination	146
Equation	16
Equations solved by factoring	290
Equivalent equations	176
fractions	118
Evaluation	8
Exponent	38
Exponents, laws of	60, 70
Expressions, algebraic	37
Factor	37, 97
Factorial	280
Formula	2, 138
Fractional exponent	198
Fractions	115, 286
Function	40
Graphs	169, 173, 227
Highest common factor	276
Homogeneous polynomial	61
Hypotenuse	188
Identity	75
Imaginary number	181, 210
Inconsistent equations	176
Indeterminate equations	145
Inverse variation	285
Inversion, proportion by	241
Irrational number	181
Joint variation	285

	PAGE
Linear equations	75, 175
Literal equations	136, 160, 211, 220
factor	37
quadratic equations	220
simultaneous equations	160
Lowest common denominator	118
multiple	278
Mean proportional	240
Member	16
Mixed expression	124
Monomial	5
Multiple	278
Multiplication	33, 59, 61, 127, 202, 210
Negative exponents	210
numbers	23, 25
Numerical equation	76, 211
factor	37
Order of operations	6
Ordinate	171
Origin	170
Parentheses	6, 7, 55, 56, 58
Plotting	171, 173, 227
Polynomial	5
Positive numbers	25
Power	5, 38, 179
Prime	97
Principal root	182
Proportion	240
Pure quadratic	211
Quadrants	170
Quadratic equation	211
trinomial	106, 288
Radical equations	209, 222
sign	181, 182
Ratio	237
Rational number	181
Rationalizing a denominator	206
Real number	181
Reciprocal	129
Right triangle	188
Root	38, 76, 181
Satisfying an equation	75
Sign of a fraction	118
Signs	34, 35, 55, 56, 60, 70, 113, 118, 170, 180, 182, 212
Similar surds	200
terms	38
Simple equations	75
Simultaneous equations	145
quadratic equations	225
Solving an equation	75
Special products and quotients	89, 96
Square	5
Square root	183, 188
Substitution	149
Subtraction	31, 47, 121, 200
Summary of factoring	113
of special products and quotients	96
Surd	192
Symbols	5
Terms	6
Terms of a fraction	115
of a proportion	240
of a ratio	237
Transposition	76
Trinomial	6
Unknown quantity	16
Variable	175
Variation	284
Zero, properties of	36
Zero exponent	210

9 1

1 12

3 4 8 3

3 3 1 4

12 36

9 3

3 33

1 3

9 27

3 x 6

3 x 2

18

6

12

Mon.	Tues.	Wed.	Thurs.	Fri.
Algebra	Study Hall	Study Hall	Study	Algebra
Piano Practice	Piano Practice	Piano Practice	Piano Practice	English
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5 minutes	English	Algebra	English	Latin
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